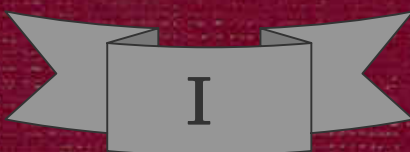


NAIL H. IBRAGIMOV

SELECTED WORKS



ALGA Publications

Nail H. Ibragimov

SELECTED WORKS

Volume I

Lie group analysis,
Differential equations,
Riemannian geometry,
Lie–Bäcklund groups,
Mathematical physics

ALGA Publications
Blekinge Institute of Technology
Karlskrona, Sweden

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Dedication

*This book is dedicated to my teachers
L.P. Barkhat and L.V. Ovsyannikov
to whom I owe, to a great extent,
my interest in mathematics.*



Figure 1: At high school I had an extraordinary teacher in mathematics, Larisa Petrovna Barkhat. Urussu, Russia, 1956.



Figure 2: Lev Vasilyevich Ovsiyanikov and I at the Symposium "Symmetry, similarity and group theoretic methods in mechanics", August 1974, Calgary, Canada.

Preface

An initial idea that led to the present collection was to have the LATEX files of my old papers for my own use, particularly of those published only in Russian. My assistant Elena D. Avdonina (Ishmakova) translated into English the papers published in Russian and also made the layout of the whole collection in LATEX. When this work was done, I decided to add current papers and publish the outcome as Selected Works in several volumes in order to make the collection available to my colleagues.

Volume I contains 21 papers. Paper 1 which opens the volume is my first research work carried out under supervision of Professor L.V. Ovsyannikov in 1964 when I was a third-year student at Novosibirsk University. Later it was published in [67] in an essentially abbreviated form. The main topics discussed in this volume are as follows.

Lie group analysis of differential equations (Papers 4, 9, 10, 11, 16, and 21). It is shown here, *inter alia*, that the Huygens principle in wave propagation has a group theoretic nature and, using this observation, Hadamard's problem is solved in space-times with nontrivial conformal group (Papers 10 and 11). Fundamental solutions and Riemann's method are studied from group point of view in Paper 21, Chapter 5.

Fluid dynamics and mathematical physics (Papers 1, 2, 5, 20). Here, one can find, in particular, an extension of Pauli's group for Dirac's equations.

Riemannian geometry (Papers 3, 6, 7). In these papers, V.A. Fock's problem on uniqueness of harmonic coordinates is solved, a theory of generalized motions in Riemannian spaces is developed and Killing's equations are generalized, the wave equation in curved space-times is defined.

Symmetry and conservation laws (Papers 8, 12, 13, 15). These papers contain a differential algebraic proof of Noether's theorem, the inverse Noether theorem, and new conservation laws in fluid dynamics.

Infinite-order tangent transformations, or Lie-Bäcklund transformation groups (Papers 14, 17, 18, 19). Theory of Lie-Bäcklund transformation groups is given in Paper 18.

Papers 1, 2 and 12 are presented here in their original unabridged versions. The original version of the concluding Paper 21 contained 184 pages typed in Russian. But it was shortened by technical reasons. In this edition,

I used the original version of § 3.3.2. Besides, I added § 5.1 and the proof of Theorem 21.18 in § 4.4, both taken from the original manuscript.

I express my gratitude to Elena for her skill in translating and preparing the book for publication.

At the final stage in the preparation of this collection, a tremendous work was done by my wife Raisa who carefully checked the whole collection and suggested a number of improvements. I am cordially grateful to Raisa for her invaluable assistance and lasting encouragement. It is a pleasure to thank my daughters Sania and Alia whose linguistic skills were a great support in all stages of my work.

My sincere thanks are due to the Vice-Chancellor of Blekinge Institute of Technology Professor Lars Haikola for his lasting support and to my colleague Associate Professor Claes Jogr  us for his assistance.

Finally, I use this opportunity to express my gratitude to my teacher Professor Lev V. Ovsyannikov to whom I owe much more than can be expressed in this short Preface.

Nail H. Ibragimov
Karlskrona,
9 May 2006

Foreword by L.V. Ovsyannikov

Professor Nail Ibragimov is an outstanding mathematician, who is well known among the international scientific community. He is one of the foremost experts in the field of group analysis of differential equations. The research in this field, initiated at the end of the nineteenth century by the works of the Norwegian mathematician Sophus Lie reached avalanche speed in the 1960s, when the importance of symmetry in the mathematical description and investigation of complex natural phenomena became clear. The solution of non-linear equations that describe these phenomena cannot be based on the standard superposition principle typical for linear problems. Here, applications of the theory of Lie groups and algebras were very fruitful. They have now been firmly established and incorporated into the arsenal of modern methods.

The multifaceted talent of Nail Ibragimov became apparent in the solution of concrete problems as well as in the wide disseminating of the methods of group analysis of differential equations. The story of his life is not trivial. After studying at the Institute of Physics and Technology in Moscow, he graduated at Novosibirsk State University and started working at the Institute of Hydrodynamics of the Siberian Branch of the USSR Academy of Sciences. He worked there in the years 1963-1980 and made significant developments in group analysis of differential equations. For these results N. Ibragimov was given the title of Doctor of Science in Physics and Mathematics and awarded the USSR State Prize in Science and Technology in 1983. Then, after the Bashkir and Moscow periods, he started his international activity. He lived and worked, first in Turkey, then in the Republic of South Africa, and since 2000, in Sweden.

The admirable industriousness and the ability to speak several languages allowed N. Ibragimov to quickly come into contact with the leading universities of many countries, where he lectured in various courses, such as mathematical analysis, group theory and Lie algebras, mathematical physics and continuum mechanics. He organized and chaired ten international conferences in Modern Group Analysis (MOGRAN). Furthermore, he compiled and published three reference books with the description of the main methods and algorithms from group analysis with a plethora of applications,



Figure 3: Prof. L.V. Ovsyannikov
At inauguration of ALGA
Karlskrona, 16 March 2005

many of which are due to his own considerable contributions.

Among the new results achieved by N. Ibragimov in the field of group analysis of differential equations, some are particularly impressive. He found an extension and the sufficient condition for inversion of the classical theorem of E. Noether concerning the existence of conservation laws for Euler's equations, obtained from functionals invariant with respect to finite Lie groups. He discovered a remarkable connection between the Huygens principle for the solutions of hyperbolic linear second-order equations and the property of conformal invariance in the Riemannian spaces associated with such equations. He also showed the efficiency of the utilization of the theory of Lie-Bäcklund groups in problems of mathematical physics and

contributed significantly to the development of this theory. Recently, he began a systematic investigation of so-called "approximate" transformations and their applications to differential equations with a "small" parameter. One of his latest results deals with the construction of the basis of differential invariants for the coefficients of second-order linear differential equations in two independent variables.

It is impossible to describe all achievement of N. Ibragimov in this short preface, and neither was such a goal set. His works (over 100) have been published in different languages in various journals, conference proceedings, etc. Therefore his wish to collect the most remarkable of them in one book is completely justified. Along with this, as a good wish to Nail Ibragimov, I would like to express my certitude that the list of his scientific achievements represented in this book will be repeatedly updated.

Professor L.V. Ovsyannikov
Member of the Russian Academy of Sciences
Novosibirsk, 29 July 2004

ПРЕДИСЛОВИЕ

Профессор Наиль Ибрагимов является выдающимся математиком, хорошо известным в мировом научном сообществе крупным специалистом в области группового анализа дифференциальных уравнений. Эта область науки, инициированная трудами норвежского математика Софуса Ли, во второй половине XVIII века, получила широкое развитие начиная с 60-х годов прошлого века, когда стала ясной важная роль симметрии в математическом описании и исследовании сложных природных процессов. Решение возникающих при этом нелинейных уравнений не может опираться на стандартный принцип суперпозиции, характерный для линейных задач. Здесь оказались плодотворными приложения теории групп и алгебр Ли, которые в настоящее время прочно вошли в арсенал применяемых методов.

Многогранный талант Наиль Ибрагимова проявился как в решении конкретных задач, так и в широкой пропаганде методов группового анализа дифференциальных уравнений. Его жизненный путь не тривиален. После учебы в Московском Физико-Техническом Институте он окончил Новосибирский Государственный Университет и поступил на работу в Институт гидродинамики Сибирского Отделения Академии Наук СССР, где в период с 1963 по 1980 г. выполнил свои первые исследования по групповому анализу дифференциальных уравнений. Полученные здесь Н. Ибрагимовым научные результаты были отмечены присуждением ему ученой степени доктора физико-математических наук и государственной премии СССР по науке и технике. Затем, после Башкирского и Московского периодов, началась его международная деятельность. Он жил и работал сначала в Турции, затем в ЮАР, а с 2000 г. в Швеции.

Завидная работоспособность и владение несколькими языками позволили Н. Ибрагимову быстро входить в контакт с ведущими университетами многих стран, где он читал разнообразные курсы лекций по математическому анализу, теории групп и алгебр Ли, математической физике, механике сплошных сред. Он организовал и провел 9 международных конференций по современному групповому анализу, а также составил и издал три справочника с описанием и многими примерами методов и алгоритмов группового анализа, в развитии которых есть и его личный весомый вклад.

Среди полученных Н. Ибрагимовым новых результатов в области группового анализа дифференциальных уравнений есть особенно впечатляющие. Он нашел усиление и достаточное условие обращения классической теоремы Э. Нётер о существовании законов сохранения для уравнений Эйлера, порождаемых инвариантным относительно конечномерной группы Ли функционалом. Им обнаружена замечательная связь принципа Гюйгенса для решений линейного уравнения второго порядка гиперболического типа со свойством конформной инвариантности риманова пространства, ассоциированного с таким уравнением. Он показал эффективность использования теории групп Ли — Беклунда в приложениях к задачам математической физики и сделал свой вклад в развитие этой теории. Сравнительно недавно им был открыт новый цикл исследования так называемых “приближенных” преобразований и их применения к дифференциальным уравнениям с “малым” параметром. Один из его последних результатов состоит в построении базиса дифференциальных инвариантов для коэффициентов линейного дифференциального уравнения второго порядка с двумя независимыми переменными.

Описать все научные результаты Н. Ибрагимова в данном кратком предисловии невозможно, да такая цель и не ставилась. Его работы (более 100) опубликованы во многих статьях, в различных журналах и сборниках, на разных языках. Поэтому представляется вполне оправданным его желание собрать наиболее яркие из них в одной книге. Вместе с тем, в качестве доброго пожелания Наилю Ибрагимову, можно выразить уверенность в том, что представленный в этом сборнике набор его научных достижений будет еще неоднократно пополнен.

Академик РАН
Л. В. Овсянников.
29 июля 2004 г.

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Paper 1

Classification of invariant solutions of the equations for two-dimensional transient-state gas flows

N. H. IBRAGIMOV

Original unabridged version of paper [67]

The dynamics of an ideal gas (a compressible fluid where $p = f(\rho, S)$) is described by the following system of non-linear differential equations:

$$\frac{d\mathbf{v}}{dt} + \frac{1}{\rho} \text{grad } p = 0, \quad \frac{d\rho}{dt} + \rho \text{div } \mathbf{v} = 0, \quad \frac{dS}{dt} = 0, \quad (1)$$

where the independent variables are the time t and the components x^i of the position vector $\mathbf{x} = (x^1, \dots, x^n)$ of *fluid particles*, while the dependent variables are the velocity $\mathbf{v} = (v^1, \dots, v^n)$, the pressure p and the density ρ of the fluid. Here $S = S(p, \rho)$ is the entropy and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \quad (2)$$

is the *particle derivative* which acts on functions $F(t, \mathbf{x})$ as follows:

$$\frac{dF}{dt} = F_t + v^i F_{x^i} \equiv F_t + (\mathbf{v} \cdot \text{grad } F),$$

where the subscripts denote the respective partial derivatives. Since n has the physical meaning of the dimension of the gas flow, it assumes the numerical values 1, 2 and 3 for one-dimensional, two-dimensional (planar) and three-dimensional flows, respectively.

Let us substitute

$$\frac{dS}{dt} = S_p \frac{dp}{dt} + S_\rho \frac{d\rho}{dt}$$

in the third equation (1), and replace $D_t(\rho)$ by $-\rho \operatorname{div} \mathbf{v}$ according to the second equation (1). Then Equations (1) become

$$\frac{d\mathbf{v}}{dt} + \frac{1}{\rho} \operatorname{grad} p = 0, \quad \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \quad \frac{dp}{dt} - \rho \frac{S_\rho}{S_p} \operatorname{div} \mathbf{v} = 0,$$

provided that $S_p \neq 0$. Using the equation (2) and denoting

$$A = -\rho \frac{S_\rho}{S_p}, \tag{3}$$

we finally obtain the following form of the system (1):

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \operatorname{grad} p &= 0, \\ \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \rho) + \rho \operatorname{div} \mathbf{v} &= 0, \\ \frac{\partial p}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} p) + A(p, \rho) \operatorname{div} \mathbf{v} &= 0, \end{aligned} \tag{4}$$

where $A(p, \rho)$ is a new arbitrary function instead of the entropy S .

Recall that a gas is said to be *polytropic* (or *perfect polytropic*) if its equation of state has the form $p = \sigma(S)\rho^\gamma$ with any function $\sigma(S)$, where γ is a constant known as a *polytropic exponent*. Hence, the entropy $S(p, \rho)$ of a polytropic gas is a function of the combination $p\rho^{-\gamma}$. According to (3) the function $A(p, \rho)$ for a polytropic gas has the form $A = \gamma p$.

We consider the system (4) in the two-dimensional ($n = 2$) case. Accordingly, we denote $\mathbf{x} = (x, y)$ and $\mathbf{v} = (u, v)$. The paper is devoted to classification of non-similar invariant planar flows of a gas obtained by using the optimal systems of one-dimensional and two-dimensional subalgebras* of the Lie algebra admitted by the equations (4). We will consider separately the arbitrary gas and the polytropic gas, i.e. the cases

I: $A(p, \rho)$ is an arbitrary function

and

II: $A = \gamma p$,

respectively.

* *Author's note to this 2006 edition:* The optimal systems are calculated in my MSc thesis. English translation of the thesis is under preparation for publication in one of the subsequent volumes of the present *Collection*.

The classification of the forms of the function $A(p, \rho)$ in accordance with symmetries (i.e. admitted groups of transformations or their Lie algebras) of the system (4) is given in [138]. According to this classification Equations (4) with an arbitrary function $A(p, \rho)$ (the case **I**) admit the seven-dimensional Lie algebra spanned by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = t \frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \\ X_6 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_7 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}. \end{aligned} \quad (5)$$

In the case **II**, the following operators are added to (5):

$$X_8 = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + 2\rho \frac{\partial}{\partial \rho}, \quad X_9 = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}. \quad (6)$$

Moreover, the group is extended further if $\gamma = 2$. Namely the following operator is added to (5) and (6):

$$\begin{aligned} X_{10} &= t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + ty \frac{\partial}{\partial y} + (x - tu) \frac{\partial}{\partial u} \\ &\quad + (y - tv) \frac{\partial}{\partial v} - 4tp \frac{\partial}{\partial p} - 2t\rho \frac{\partial}{\partial \rho}. \end{aligned} \quad (7)$$

Let us denote the admitted group in the case **I** by G_7 , while the groups in the case **II** are denoted by G_9 and G_{10} for an arbitrary γ and for $\gamma = 2$, respectively. The respective Lie algebras are denoted by L_7, L_9, L_{10} .

Table 1.1 contains the optimal system of one-dimensional subalgebras of L_9 , and hence furnishes the optimal system of one-parameter subgroups for the group G_9 . When $\delta = 0$, the optimal system of one-parameter subgroups for the group G_7 is generated by the operators 1 – 7 from Table 1.1 and the operator $X_4 + \alpha X_6$. The similar optimal system for the group G_{10} is generated by the operators 1 – 12 from Table 1.1 and the operators

$$\begin{aligned} 14. \quad & X_1 + X_2 + X_7 + \delta X_9 + X_{10}, \\ 15. \quad & X_1 + \alpha X_6 + \beta X_7 + \delta X_9 + X_{10}. \end{aligned} \quad (8)$$

The parameters α and β in the operator 15 from (8) satisfy the condition $0 \leq \alpha < 2$, and $\beta \geq 0$ when $\alpha = 0$.

Tables 1.2 and 1.4 provide the optimal systems of two-parameter subgroups for the groups G_7 and G_9 , respectively. Table 1.3 (involving X_{10}) together with the subgroups 1 – 41 of Table 1.4 (not involving X_{10}) furnish the optimal system of two-parameter subgroups for the group G_{10} .

Table 1.1: One-dimensional subalgebras of L_9

1	$X_1 + \delta X_9$	$(\delta = 0; 1)$	8	$X_2 + X_8 + \delta X_9$
2	$X_2 + \delta X_9$	$(\delta = 0; 1)$	9	$X_4 + X_6 + \delta X_9$
3	$X_1 + X_4 + \delta X_9$	$(\delta = 0; 1)$	10	$\alpha X_6 + \beta X_7 + X_8 + \delta X_9$
4	$X_2 + X_5 + \delta X_9$	$(\delta = 0; 1)$	11	$X_1 - X_6 + \alpha X_7 + X_8 + \delta X_9$
5	$X_6 + \alpha X_7 + \delta X_9$		12	X_9
6	$X_7 + \delta X_9$		13	$X_5 + \delta X_9$
7	$X_1 + X_7 + \delta X_9$			

Table 1.2: Two-dimensional subalgebras of L_7

1	X_1, X_2	6	$X_5, \alpha X_4 + X_6$	10	$X_2, X_1 + X_4 + \alpha X_5$
2	X_1, X_6	7	X_7, X_6	11	$X_2, \alpha X_4 + \beta X_5 + X_6$
3	$X_1, \alpha X_6 + X_7$	8	X_2, X_3	12	X_2, X_4
4	X_5, X_4	9	$X_2, X_3 + X_4$	13	$X_2, X_1 + X_5$
5	$X_5, X_2 + \alpha X_3$			14	$X_2 + X_5, \alpha X_2 + \beta X_3 + X_4$

Table 1.3: Special two-dimensional subalgebras of L_{10}

1	$X_9,$	$X_1 + X_2 + X_7 + X_{10}$
2	$X_9,$	$X_1 + \alpha X_6 + \beta X_7 + X_{10} \quad (0 \leq \alpha < 2)$
3	$X_2 + X_5,$	$X_1 + \alpha(X_6 - X_8) - X_7 + \beta X_9 + X_{10} \quad (\alpha \neq 0)$
4	$X_2 + X_5,$	$X_1 + \alpha X_2 + \beta X_3 + \delta X_9 + X_{10} - X_7$
5	$X_2 + X_5 + X_9,$	$X_1 + \alpha X_2 + \beta X_3 - X_7 + X_{10}$
6	$X_7 + \delta X_9,$	$X_1 + \alpha X_6 + \beta X_9 + X_{10} \quad (0 \leq \alpha < 2)$
7	$X_6 + \varepsilon X_7 - X_8 + \delta X_9,$	$X_1 + \alpha X_6 + \beta X_7 + \chi X_9 + X_{10} \quad (0 \leq \alpha < 2)$

Table 1.4: Two-dimensional subalgebras of L_9

1	$X_1 + X_9, \quad \alpha X_1 + X_7$	26	$X_1, \quad X_7 + \alpha X_8 + \beta X_9$
2	$X_1 + X_9, \quad \alpha X_1 + X_6 + \beta X_7 - X_8$	27	$X_1, \quad X_6 + \alpha X_7 + \beta X_8 + \delta X_9$
3	$X_2 + X_9, \quad X_3 + X_4$	28	$X_1, \quad X_9$
4	$X_2 + X_9, \quad X_1 + \alpha X_4 + \beta X_5$	29	$X_2, \quad X_1 + X_4 + \beta X_5 + \delta X_9 \quad (\delta = 0; 1)$
5	$X_2 + X_9, \quad \alpha X_2 + \beta X_4 + X_5 \quad (\alpha = 0; 1)$	30	$X_2, \quad X_1 + \beta X_5 + \delta X_9 \quad (\beta, \delta = 0; 1)$
6	$X_2 + X_9, \quad \alpha X_2 + \beta X_3 + X_8$	31	$X_2, \quad \alpha X_4 + X_5 + \beta X_9 \quad (\beta = 0; 1)$
7	$X_2 + X_5, \quad \alpha X_2 + X_4 + \beta X_3 + \delta X_9$ $(\delta = 0; 1)$	32	$X_2, \quad X_3 + X_4 + \alpha X_9 \quad (\alpha = 0; 1)$
		33	$X_2, \quad X_3 + \alpha X_9 \quad (\alpha = 0; 1)$
8	$X_7 + \alpha X_9, \quad X_6 + \beta X_9$	34	$X_2, \quad X_4 + \alpha X_9 \quad (\alpha = 0; 1)$
9	$X_7 + \alpha X_9, \quad \beta X_6 + X_8 + \delta X_9$	35	$X_2, \quad X_4 + \alpha X_5 + X_6 + \beta X_9$
10	$X_1 + X_4, \quad -2X_6 + X_8 + \alpha X_9$	36	$X_2, \quad X_5 + X_6 + \alpha X_9$
11	$X_2 + X_5, \quad -X_6 + X_8 + \alpha X_9$	37	$X_2, \quad X_6 + \alpha X_9$
12	$X_7 + \alpha X_9, \quad X_1 - X_6 + X_8 + \beta X_9$	38	$X_2, \quad X_3 + X_8 + \alpha X_9$
13	$X_1 + X_7 + \alpha X_9, \beta X_1 - X_6 + X_8 + \delta X_9$	39	$X_2, \quad \alpha X_6 + X_8 + \beta X_9$
14	$X_6 + \alpha X_7 + \beta X_9, \quad \delta X_7 + X_8 + \varepsilon X_9$	40	$X_2, \quad X_1 - X_6 + X_8 + \alpha X_9$
15	$X_9, \quad X_1 + X_4$	41	$X_2, \quad X_9$
16	$X_9, \quad X_2 + X_5$	42	$X_5, \quad X_3 + X_9$
17	$X_9, \quad X_6 + \alpha X_7$	43	$X_5, X_2 + \alpha \beta X_3 + X_4 + \beta X_9 \quad (\beta = 0; 1)$
18	$X_9, \quad X_7$	44	$X_5, \quad X_3 + X_4 + X_9$
19	$X_9, \quad X_1 + X_7$	45	$X_5, \quad X_4 + \alpha X_9 \quad (\alpha = 0; 1)$
20	$X_9, \quad X_2 + X_8$	46	$X_5, \quad X_3 + X_8 + \alpha X_9$
21	$X_9, \quad X_4 + X_6$	47	$X_5, \quad \alpha X_4 + X_6 + \beta X_9 \quad (\alpha = 0; 1)$
22	$X_9, \quad \alpha X_6 + \beta X_7 + X_8$	48	$X_5, \quad X_2 + \alpha X_3 + X_8 + \beta X_9$
23	$X_9, \quad X_1 - X_6 + \alpha X_7 + X_8$	49	$X_5, \quad \alpha X_6 + X_8 + \beta X_9$
24	$X_1, \quad X_8 + \alpha X_9$	50	$X_5, \quad X_9$
25	$X_1, \quad X_2 + X_8 + \alpha X_9$	51	$X_5 + X_9, \quad \alpha X_4 + \beta X_5 + X_6$

Let us find the general form of the invariant solutions of the rank 1. They are derived from two-parameter subgroups. The values U, V, P, R depend on a single argument λ . The expressions of λ in terms of t, x, y vary with the subgroup and are indicated below. The necessary condition of existence of the invariant solution is not satisfied for the subgroups where X_9 is one of the basic generators. Moreover, presence of X_9 in generators of every subgroup affects only the form of p and ρ and this influence can be easily taken into account. Therefore, we ignore the operator X_9 . The subgroups having X_1, X_2, X_3, X_4 or X_5 as one of the basic generators will also be omitted from our consideration, for they provide the stationary and one-dimensional cases. For instance, the invariant solution for $H = \langle X_5 \rangle$ has the form

$$u = U(t, x), \quad v = \frac{y}{t} + V(t, x), \quad p = \frac{1}{t}P(t, x), \quad \rho = \frac{1}{t}R(t, x),$$

and substitution in gasdynamic equations yields the following system for the invariant solution:

$$\begin{aligned} V_t + UV_x + t^{-1}V &= 0, & U_t + UU_x + R^{-1}P_x &= 0, \\ R_t + UR_x + RU_x &= 0, & P_t + UP_x + A'U_x &= 0 \\ & \left(A' = -R \frac{\partial S / \partial R}{\partial S / \partial P} \right). \end{aligned}$$

Thus, U, P, R satisfy the one-dimensional gasdynamic equations whereas V is obtained from the equation

$$V_t + UV_x + t^{-1}V = 0$$

with a known function $U(t, x)$. Hence, one has to solve only the equations for a one-dimensional gas motion.

The subgroups of Table 1.2 furnish the invariant solutions of the form

$$7. \quad u_r = U, \quad u_\varphi = V, \quad p = P, \quad \rho = R; \quad \lambda = \frac{r}{t},$$

where r, φ are the polar coordinates on the (x, y) plane while u_r, u_φ are the projections of the velocity on the axes of the polar coordinates, and

$$14. \quad u = \frac{tx - y}{t^2 + \alpha t - \beta} + U, \quad v = \frac{(\alpha + t)y - \beta x}{t^2 + \alpha t - \beta} + V, \quad p = P, \quad \rho = R; \quad \lambda = t.$$

For Table 1.4, one needs to consider only the subgroups 9 – 14. We have

$$9. \quad u_r = r^{-1/\beta}U, \quad u_\varphi = r^{-1/\beta}V, \quad p = P, \quad \rho = r^{2/\beta}R; \quad \lambda = tr^{-(\beta+1)/\beta} \text{ when } \beta \neq 0,$$

$$u_r = t^{-1}U, \quad u_\varphi = t^{-1}V, \quad p = P, \quad \rho = t^2R; \quad \lambda = r \text{ when } \beta = 0$$

$$10. \quad u = t + \sqrt{y}U, \quad v = \sqrt{y}V, \quad p = P, \quad \rho = y^{-1}R; \quad \lambda = (1/2) t^2y^{-1} - xy^{-1}$$

$$11. \quad u = (tx - y)U, \quad v = x + (tx - y)V, \quad p = P, \quad \rho = (tx - y)^{-2}R; \quad \lambda = t$$

$$12. \quad u_r = rU, \quad u_\varphi = rV, \quad p = P, \quad \rho = r^{-2}R; \quad \lambda = re^t$$

$$13. \quad u_r = rU, \quad u_\varphi = rV, \quad p = P, \quad \rho = r^{-2}R; \quad \lambda = t + \varphi + \beta \ln r$$

$$14. \quad u_r = r^{\alpha/\delta} e^{\varphi/\delta} U, \quad u_\varphi = r^{\alpha/\delta} e^{\varphi/\delta} V, \quad p = P, \quad \rho = t^2 r^{-2} R; \quad \lambda = r^{\alpha-\delta} t^\delta e^\varphi,$$

when $\delta \neq 0$, and

$$u_r = rt^{-1}U, \quad u_\varphi = rt^{-1}V, \quad p = P, \quad \rho = t^2 r^{-2} R; \quad \lambda = \varphi + \alpha \ln r$$

when $\delta = 0$.

For Table 1.3, one has the following representations of invariant solutions:

$$3. \quad u = \frac{e^{\theta(t)}}{1+t^2}(U - tV + \lambda t^2), \quad v = x + \frac{e^{\theta(t)}}{1+t^2}[V + tU - \lambda t(t^2 + 2)],$$

$$p = \frac{P}{(1+t^2)^2}, \quad \rho = \frac{e^{2\theta(t)}R}{(1+t^2)}; \quad \lambda = \frac{(tx - y)e^{-\theta(t)}}{1+t^2}.$$

$$4. \quad u = \frac{U - tV + \lambda t^2 + (1/2)\beta t}{1+t^2} - \frac{\beta\theta(t)}{2\alpha}, \quad p = \frac{P}{(1+t^2)^2},$$

$$v = x + \frac{V + tU - \lambda t(t^2 + 2) + (1/2)\beta t^2}{1+t^2} + \frac{\beta t\theta(t)}{2\alpha}, \quad \rho = \frac{R}{1+t^2};$$

$$\lambda = \frac{tx - y + (1/2)\alpha + (1/2)\beta t}{1+t^2} + \frac{\beta}{2\alpha}\theta(t), \quad \theta(t) = \alpha \arctan t.$$

6. For the sake of simplicity let us consider the case $\alpha = \beta = 0$. Then

$$u_r = \frac{rt}{1+t^2} + \frac{U}{r}, \quad u_\varphi = \frac{V}{r}, \quad p = \frac{P}{(1+t^2)^2}, \quad \rho = \frac{R}{1+t^2}; \quad \lambda = \frac{r}{\sqrt{1+t^2}}.$$

7. Let us consider the case $\alpha = \beta = 0$ again. Then

$$u_r = \frac{rt}{1+t^2} + \frac{rU}{1+t^2}, \quad u_\varphi = \frac{rV}{1+t^2}, \quad p = \frac{P}{(1+t^2)^2},$$

$$\rho = \frac{R}{r^2}; \quad \lambda = \varphi + \delta \ln \frac{r}{\sqrt{1+t^2}}.$$

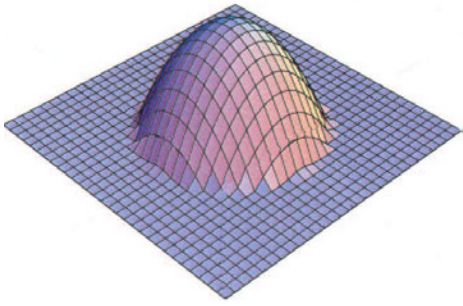


Figure 1: Flow of a hill of water.

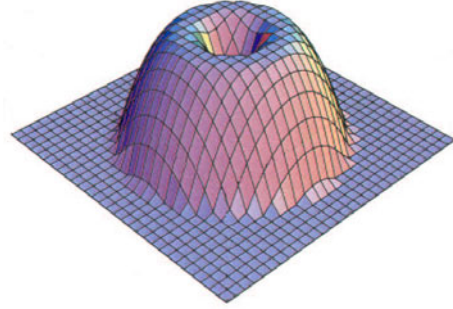


Figure 2: Flow of an annular mass of water.

The invariant solutions of the rank 2 are obtained likewise. For example, consider the invariant solution for $H = \langle X_7, X_1 + X_{10} \rangle$. It has the form

$$u_r = \frac{rt}{1+t^2} + \frac{U}{r}, \quad u_\varphi = \frac{V}{r}, \quad p = \frac{P}{(1+t^2)^2}, \quad \rho = \frac{R}{1+t^2}; \quad \lambda = \frac{r}{\sqrt{1+t^2}}. \quad (9)$$

In the given case $\gamma = 2$, the equations of gas dynamics can be treated as shallow water equations. One can assume without loss of generality that density of water and acceleration due to gravity are equal to one. Then $p = \rho^2/2$, and ρ represents the height of the water above the even base. We shall consider water motion over a dry base. Substituting (9) into (4), written in the polar coordinates, one obtains one of the solutions

$$U = 0, \quad V = 0, \quad R = \frac{1}{2}(a^2 - \lambda^2); \quad \lambda \leq a, \quad a = \text{const.}$$

Thus, the invariant solution is

$$u_r = \frac{rt}{1+t^2}, \quad u_\varphi = 0, \quad \rho = \frac{1}{2(1+t^2)} \left(a^2 - \frac{r^2}{1+t^2} \right).$$

It describes a flow of a hill of water over a dry flat base (Fig. 1). Initially ($t = 0$) the hill of water is at rest and has the form

$$\rho = (a^2 - r^2)/2, \quad r \leq a.$$

The boundary ($\rho = 0$) moves according to the law $r = a\sqrt{1+t^2}$ and the height of the peak of the hill, located at $r = 0$, decreases in time by the law $a^2/[2(1+t^2)]$. The velocity of the flow remains bounded, $u_r < a$. It should be noted that in this solution the velocity is a linear function of the coordinates and that it is contained in the class of solutions obtained by L.V. Ovsyannikov in [135].

Let us consider an invariant solution describing an interesting motion in channels in the framework of shallow-water theory*. We will write the differential equations of the shallow water theory in physical notation, namely:

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + g \nabla h &= 0, \\ h_t + \mathbf{v} \cdot \nabla h + h \operatorname{div} \mathbf{v} &= 0. \end{aligned} \quad (10)$$

Here h is the height of the water surface above the base, g is the constant of acceleration of gravity, $\mathbf{v} = (v_1, v_2)$. Let us consider again the invariant solutions under the subalgebra L_2 spanned by

$$\begin{aligned} X_7 &= x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + v^2 \frac{\partial}{\partial v^1} - v^1 \frac{\partial}{\partial v^2}, \\ X_1 + X_{10} &= (1 + t^2) \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} + (x^i - tv^i) \frac{\partial}{\partial v^i} - 2th \frac{\partial}{\partial h}. \end{aligned}$$

The reckoning shows that the invariant solution has the form

$$\begin{aligned} v^1 &= \frac{tx^1}{1+t^2} + \frac{1}{r^2} [x^1 U(\lambda) - x^2 V(\lambda)], & h &= \frac{H(\lambda)}{1+t^2}, \\ v^2 &= \frac{tx^2}{1+t^2} + \frac{1}{r^2} [x^2 U(\lambda) + x^1 V(\lambda)]; & \lambda &= \frac{r}{\sqrt{1+t^2}} \end{aligned} \quad (11)$$

and is defined by the following system of ordinary differential equations:

$$\lambda U U' - U^2 - V^2 + g \lambda^3 H' + \lambda^4 = 0, \quad UV' = 0, \quad (UH)' = 0. \quad (12)$$

Equations (12) describe various rotationally symmetric flows of shallow water over a flat bottom. One of them is obtained by setting $U = 0$, $V = K\lambda$ and integrating (12). Then (11) yields:

$$v^1 = \frac{tx^1}{1+t^2} - K \frac{x^2}{r\sqrt{1+t^2}}, \quad v^2 = \frac{tx^2}{1+t^2} + K \frac{x^1}{r\sqrt{1+t^2}}$$

for the velocity, and

$$h = \frac{1}{(1+t^2)g} \left[C - \frac{r^2}{2(1+t^2)} + K^2 \ln \frac{r}{\sqrt{1+t^2}} \right], \quad K, C = \text{const.} \quad (13)$$

for the height of the water above a flat bottom.

Figure 2 describes the motion of a bounded annular mass of water with the initial ($t = 0$) radii $r_1(0) > 0$ and $r_2(0) > r_1(0)$ of the boundary circles

* *Author's note to this 2006 edition:* This solution was obtained in my MSc work but was not included either in the final text of the MSc thesis or in paper [67].

and the initial angular velocity $\omega(0) = K/r$. As the water spreads under the action of gravity and rotation, the angular velocity ω decays as

$$\omega = \frac{K}{r\sqrt{1+t^2}}.$$

The bottom boundary ($h = 0$) of the water consists of two concentric circles with the radii $r_1 = r_1(0)\sqrt{1+t^2}$ and $r_2 = r_2(0)\sqrt{1+t^2}$.

I am indebted to Professor L.V. Ovsyannikov for helpful advice.

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Paper 2

Group properties of wave equations for particles with zero mass

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Original unabridged version of paper [70]

Introduction

The conformal invariance of the Dirac equations for particles with zero mass (neutrinos) and the Maxwell equations in vacuum was established in works [36], [16], [39], [129]. However, to the best of my knowledge, it has not been demonstrated anywhere that the conformal group is the widest possible group admitted by these equations. Furthermore, the group properties of the linear equations of weak gravitational waves (gravitons) derived in the recent work of G. Rumer [157] were not studied yet.

In the present paper I investigate the symmetries of wave equations for particles with zero mass: neutrinos, photons and gravitons. Specifically, the solution to the problem of finding the widest symmetry groups of point transformations is given for the massless Dirac equations, the Maxwell equations in vacuum and Rumer's equations.

We will use only the lower indices and assume that the summation is taken over all repeated indices. The subscripts k, l, m, n run through the values 1, 2, 3, 4, and the subscripts p, q, r, v run through the values 1, 2, 3. The permutation symbols (unit skew-symmetric tensors) in three and four dimensional spaces are denoted by e_{pqr} , and e_{klmn} , respectively. By definition, one has $e_{123} = 1$, $e_{213} = -1$, $e_{112} = 0$, and likewise $e_{1234} = 1$, etc.

Generators of transformation groups in the four-dimensional Euclidean space $E_4(x)$ are written in the form

$$X = \xi_k(x) \frac{\partial}{\partial x_k}, \quad (1)$$

and generators of transformation groups in the space $E_{4+N}(x, u)$, where u_τ ($\tau = 1, \dots, N$) are functions of x , are written

$$\overline{X} = \xi_k(x, u) \frac{\partial}{\partial x_k} + \eta_\tau(x, u) \frac{\partial}{\partial u_\tau}. \quad (2)$$

Applying the usual procedure for calculating symmetries for systems of differential equations [138] one can show that the operator (2) admitted by the equations considered in this paper has the special form

$$\overline{X} = X + s_{\tau\sigma}(x) u_\sigma \frac{\partial}{\partial u_\tau}, \quad (3)$$

where X has the form (1) and its coordinates ξ_k satisfy the equations

$$\frac{\partial \xi_k}{\partial x_l} + \frac{\partial \xi_l}{\partial x_k} = \mu(x) \delta_{kl}. \quad (4)$$

Equations (4) define the 15-parameter group of conformal transformations in the Minkowsky space [46]. Solving the system (4), one obtains the following basic generators of the conformal group:

$$\begin{aligned} X_k &= \frac{\partial}{\partial x_k}, \\ X_{kl} &= x_l \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_l} \quad (k < l), \\ T &= x_k \frac{\partial}{\partial x_k}, \\ Y_k &= (2x_k x_l - |x|^2 \delta_{kl}) \frac{\partial}{\partial x_l}, \end{aligned} \quad (5)$$

where $|x|^2 = \sum_k x_k^2$. Introducing the matrix $S = \|s_{\sigma\tau}\|$ and noting that

$$\overline{X} u_\tau = s_{\tau\sigma} u_\sigma = (Su)_\tau,$$

one can write the system (3) in the following alternative form:

$$\overline{X} = X + S. \quad (6)$$

§ 1 The Dirac equations

The Dirac equations for particles with the zero mass have the form

$$\gamma_k \frac{\partial \psi}{\partial x_k} = 0, \quad (7)$$

where ψ is a four-dimensional complex valued column vector:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

and γ_k are the complex valued Dirac matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Solution of the determining equations gives the following result.

Theorem 2.1. The general group of point transformations admitted by the Dirac equations (7) is the 17-parameter Lie group* with the following generators written in the form (6):

$$\begin{aligned} \overline{X} &= X + \frac{1}{8} \frac{\partial \xi_k}{\partial x_l} (\gamma_k \gamma_l - \gamma_l \gamma_k - 3\delta_{kl}), \\ \overline{A} &= \gamma_1 \gamma_2 \gamma_3 \gamma_4, \\ \overline{B} &= I. \end{aligned} \quad (8)$$

*In fact, equations (7) admit an infinite group due to linearity. Namely, all transformations of the form $\psi' = \psi + \varphi(x)$ are admitted, where $\varphi(x)$ is any solution of (7). Such transformations are not taken into consideration. This remark holds for the remaining two equations as well.

Here X runs over the system of the generators (5) of the conformal group, and I is the 4×4 identity matrix generating the group of dilations of the function ψ . When the operator X corresponds to the non-homogeneous Lorentz group, the first operator in (8) reduces to the following known formula [173]:

$$\overline{X} = X + \frac{1}{8} \frac{\partial \xi_k}{\partial x_l} (\gamma_k \gamma_l - \gamma_l \gamma_k).$$

§ 2 The Maxwell equations

Propagation of electromagnetic waves in vacuum is governed by the Maxwell equations

$$\begin{aligned} \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \text{curl } \mathbf{H}, \quad \text{div } \mathbf{E} = 0, \\ \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} &= -\text{curl } \mathbf{E}, \quad \text{div } \mathbf{H} = 0, \end{aligned} \quad (9)$$

where the “curl” and “div” are taken with respect to the spatial coordinates x, y, z . Let us set $c = 1$, denote

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = it,$$

where $i = \sqrt{-1}$, and use the skew-symmetric tensor F_{kl} (i.e. $F_{lk} = -F_{kl}$) of the electromagnetic field. The diagonal components of the tensor F_{kl} vanish (since it is skew-symmetric), i.e.

$$F_{11} = F_{22} = F_{33} = F_{44} = 0,$$

and the six independent components F_{kl} ($k < l$) are expressed via the magnetic field $\mathbf{H} = (H_x, H_y, H_z)$ and the electric field $\mathbf{E} = (E_x, E_y, E_z)$ as follows:

$$\begin{aligned} F_{12} &= H_z, & F_{13} &= -H_y, & F_{14} &= -iE_x, \\ F_{23} &= H_x, & F_{24} &= -iE_y, & F_{34} &= -iE_z. \end{aligned}$$

Thus, the tensor F_{kl} can be written as the following skew-symmetric matrix:

$$(F_{kl}) = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}.$$

In this four-dimensional tensor notation, the Maxwell equations are written:

$$\begin{aligned} \frac{\partial F_{kl}}{\partial x_m} + \frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} &= 0, \\ \frac{\partial F_{kl}}{\partial x_l} &= 0. \end{aligned} \quad (10)$$

The following equivalent form of the system (10) manifests that the first line of the system contains precisely *four* independent equations:

$$e_{klmn} \frac{\partial F_{mn}}{\partial x_l} = 0, \quad \frac{\partial F_{kl}}{\partial x_l} = 0 \quad (k = 1, \dots, 4). \quad (10')$$

Solution of the determining equations gives the following result*.

Theorem 2.2. The general group of point transformations admitted by the Maxwell equations (10) is the 17-parameter Lie group with the following generators written in the form (3):

$$\bar{X} = X - \sum_{k < l} \left(F_{km} \frac{\partial \xi_m}{\partial x_l} + F_{ml} \frac{\partial \xi_m}{\partial x_k} \right) \frac{\partial}{\partial F_{kl}}, \quad (11)$$

$$\bar{A} = \sum_{k < l} \tilde{F}_{kl} \frac{\partial}{\partial F_{kl}}, \quad \bar{B} = \sum_{k < l} F_{kl} \frac{\partial}{\partial F_{kl}}. \quad (12)$$

Here \tilde{F}_{kl} is the dual tensor to the tensor of the electro-magnetic field F_{kl} . It is defined by

$$\tilde{F}_{kl} = \frac{i}{2} e_{klmn} F_{mn}$$

and provides the following skew-symmetric matrix:

$$\left(\tilde{F}_{kl} \right) = \begin{pmatrix} 0 & E_z & -E_y & iH_x \\ -E_z & 0 & E_x & iH_y \\ E_y & -E_x & 0 & iH_z \\ -iH_x & -iH_y & -iH_z & 0 \end{pmatrix}.$$

The operator X in (11) runs over the system of the generators (5) of the conformal group.

*Yu.A. Danilov informed me that he considered group properties of Maxwell equations as well.

The formula (11) shows that F_{kl} is a tensor of the second order with respect to the conformal transformations. It means that the transformation law of the quantities under the conformal transformations $x'_k = x'_k(x)$ is given by the equations

$$F'_{kl} = F_{mn} \frac{\partial x_m}{\partial x'_k} \frac{\partial x_n}{\partial x'_l}. \quad (13)$$

Indeed, taking the infinitesimal conformal transformations and using (13), one obtains (11).

The operator \bar{B} in (12) generates the group of dilations

$$F'_{kl} = e^a F_{kl}.$$

The group generated by \bar{A} in (12) is less trivial. To manifest its physical significance, let us rewrite the operator \bar{A} in terms of the magnetic and electric vector fields, using the expressions for F_{kl} and \tilde{F}_{kl} . We have:

$$\bar{A} = E_x \frac{\partial}{\partial H_x} + E_y \frac{\partial}{\partial H_y} + E_z \frac{\partial}{\partial H_z} - H_x \frac{\partial}{\partial E_x} - H_y \frac{\partial}{\partial E_y} - H_z \frac{\partial}{\partial E_z}$$

or in the vector notation:

$$\bar{A} = \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{H}} - \mathbf{H} \cdot \frac{\partial}{\partial \mathbf{E}}.$$

It is clear now that \bar{A} is the generator of the group of rotations in the space of the magnetic and electric fields:

$$\mathbf{H}' = \mathbf{H} \cos a + \mathbf{E} \sin a, \quad \mathbf{E}' = \mathbf{E} \cos a - \mathbf{H} \sin a.$$

Let X run over the generators of the non-homogeneous Lorentz group. Solving the equations $\bar{X}(J) = 0$ for the corresponding operators (11), one obtains two functionally independent invariants

$$J_1 = \sum_{k < l} F_{kl}^2, \quad J_2 = \sum_{k < l} F_{kl} \tilde{F}_{kl}.$$

The general 17-parameter symmetry group has no invariants, i.e. it is transitive.

§ 3 Equations of a weak gravitational field

A theory of curvature waves is developed in [157] in the framework of the general relativity. It is shown there that, in the linear approximation (weak

gravitational field), the curvature waves in empty space-times ($R_{kl} = 0$) can be described in terms of two traceless symmetric tensors H_{pq} and E_{pq} by the following system of first-order partial differential equations:

$$e_{pqr} \frac{\partial H_{vr}}{\partial x_q} - \frac{\partial E_{pv}}{\partial t} = 0, \quad \frac{\partial H_{pq}}{\partial x_q} = 0, \quad (14)$$

$$e_{pqr} \frac{\partial E_{vr}}{\partial x_q} + \frac{\partial H_{pv}}{\partial t} = 0, \quad \frac{\partial E_{pq}}{\partial x_q} = 0. \quad (15)$$

These equations are quite similar to the Maxwell equations (9) for electromagnetic waves.

I will write equations (14)–(15) in the following form:

$$e_{pqr} \frac{\partial \Phi_{vr}}{\partial x_q} - \frac{\partial \Phi_{pv}}{\partial x_4} = 0, \quad \frac{\partial \Phi_{pq}}{\partial x_q} = 0, \quad (16)$$

where $x_4 = it$ is the complex time and $\Phi_{pq} = H_{pq} + iE_{pq}$ are the elements of the traceless symmetric 3×3 complex matrix (tensor of the second order). Let us take the independent elements

$$u_1 = \Phi_{11}, \quad u_2 = \Phi_{12}, \quad u_3 = \Phi_{13}, \quad u_4 = \Phi_{22}, \quad u_5 = \Phi_{23}$$

of the matrix Φ_{pq} as the dependent variables and introduce the matrices

$$\beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & -2 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

In this notation, the main result on symmetries of equations (16) is formulated as follows.

Theorem 2.3. The equations of weak gravitational waves are conformally invariant. Namely, the general group of point transformations admitted by

equations (16) is the 16-parameter Lie group with the following generators written in the form (6):

$$\begin{aligned}
\overline{X}_k &= \frac{\partial}{\partial x_k} \quad (k = 1, \dots, 4), \\
\overline{X}_{12} &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + \beta_3, \\
\overline{X}_{31} &= x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} + \beta_2, \\
\overline{X}_{23} &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + \beta_1, \\
\overline{X}_{41} &= x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1} + \beta_1, \\
\overline{X}_{42} &= x_2 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_2} + \beta_2, \\
\overline{X}_{43} &= x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3} + \beta_3, \\
\overline{Y}_1 &= (2x_1x_k - |x|^2\delta_{1k}) \frac{\partial}{\partial x_k} + 2(x_3\beta_2 - x_2\beta_3 + x_4\beta_1 - 3x_1I), \\
\overline{Y}_2 &= (2x_2x_k - |x|^2\delta_{2k}) \frac{\partial}{\partial x_k} + 2(x_1\beta_3 - x_3\beta_1 + x_4\beta_2 - 3x_2I), \\
\overline{Y}_3 &= (2x_3x_k - |x|^2\delta_{3k}) \frac{\partial}{\partial x_k} + 2(x_2\beta_1 - x_1\beta_2 + x_4\beta_3 - 3x_3I), \\
\overline{Y}_4 &= (2x_4x_k - |x|^2\delta_{4k}) \frac{\partial}{\partial x_k} - 2(x_1\beta_1 + x_2\beta_2 + x_3\beta_3 + 3x_4I), \\
\overline{T} &= x_k \frac{\partial}{\partial x_k}, \\
\overline{B} &= I.
\end{aligned} \tag{17}$$

I express my gratitude to Professor L.V. Ovsyannikov for stimulating critical remarks.

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Translated by E.D. Avdonina

Paper 3

Transformations preserving harmonic coordinates

N.H. IBRAGIMOV [72]

Doklady Akademii Nauk SSSR, Tom 181, (1968), No. 5, pp. 1050–1053.

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In the article an investigation is made of the transformations taking an arbitrary harmonic coordinate system in a Riemannian space into a harmonic coordinate system. It is shown that if these transformations form a continuous group, then only the linear transformations possess the property of taking an arbitrary harmonic coordinate system into a harmonic system. As distinguished from [50], we discuss harmonic coordinates locally, without considering conditions at infinity. The discussion is conducted by a well-known method of the group properties of differential equations [138].

We shall denote by V_n an arbitrary n -dimensional Riemannian space. A system of coordinates $\{x\}$ in V_n will be called harmonic if in these coordinates the components of the metric tensor of the space V_n satisfy the equations

$$\frac{\partial}{\partial x^k}(\sqrt{|g|}g^{ik}) = 0 \quad (i = 1, \dots, n). \quad (1)$$

In any Riemann space there exists an infinite number of the harmonic coordinates. We shall denote by Γ the set of all harmonic coordinates of the space V_n and introduce the following

Definition 3.1. The transformation

$$x'^i = \varphi^i(x) \quad (i = 1, \dots, n) \quad (2)$$

is called a *transformation preserving harmonic coordinates* in V_n if it maps the set Γ into itself.

In this definition it is considered that x^i denote arbitrary harmonic coordinates so that all harmonic coordinates transform by means of the same functions ϕ^i ($i = 1, \dots, n$). In contrast to the definition introduced here, in [50] by a transformation preserving harmonic coordinates in the space V_n is understood that coordinate transformation which maps a fixed harmonic coordinate system into a harmonic one. In this sense a problem of the uniqueness of the harmonic coordinates was discussed in [52]. In what follows, we shall use the definition of the transformations preserving harmonic coordinates introduced here. We shall impose the following restrictions on the transformations (2) preserving the harmonic coordinates in the space V_n . We require that they form a continuous group H composed by one-parameter continuous local Lie groups.

Let us denote by $E(x, g)$ the Euclidean space whose points have coordinates x^i ($i = 1, \dots, n$) and g^{ik} ($i, k = 1, \dots, n$). By virtue of the formula for the transformation of the components of the metric tensor

$$g'^{ik} = g^{mn} \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \quad (i, k = 1, \dots, n), \quad (3)$$

the continuous group H of transformations (2) of the coordinates defines a continuous group of the transformations of the space $E(x, g)$ into itself, which we shall denote by \overline{H} .

Theorem 3.1. For any Riemannian space V_n the broadest continuous group H of transformations (2) preserving harmonic coordinates in V_n is of order $n(n+1)$ and consists of all linear transformations

$$x'^i = a_k^i x^k + b^i \quad (i = 1, \dots, n), \quad (4)$$

where a_k^i, b^i ($i, k = 1, \dots, n$) are arbitrary constants.

Proof of the theorem is based on the following two lemmas.

Lemma 3.1. The group \overline{H} corresponding to the linear transformations (4) is the broadest group of continuous transformations (2)-(3) admitted by equations (1) in the sense of S. Lie.

Proof. Introducing the quantities

$$h^{ik} = \sqrt{|g|} g^{ik} \quad (i, k = 1, \dots, n)$$

and setting

$$\frac{\partial h^{ik}}{\partial x^j} \equiv h_{,j}^{ik} \quad (i, k, j = 1, \dots, n),$$

we rewrite equations (1) in the form

$$h_{,k}^{ik} = 0 \quad (i = 1, \dots, n). \quad (5)$$

Then (3) yields the transformations

$$J \left(\frac{\partial x'}{\partial x} \right) h^{ik} = h^{mn} \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \quad (i, k = 1, \dots, n), \quad (6)$$

where $J(\frac{\partial x'}{\partial x})$ is the Jacobian of the coordinate transformation (2). An arbitrary generator of the group \bar{H} is written in the form

$$X = \xi^i(x) \frac{\partial}{\partial x^i} + \eta^{ik}(x, h) \frac{\partial}{\partial h^{ik}}. \quad (7)$$

Considering the infinitesimal transformation (6), one obtains

$$\eta^{ik}(x, h) = h^{il} \frac{\partial \xi^k}{\partial x^l} + h^{kl} \frac{\partial \xi^i}{\partial x^l} - h^{ik} \frac{\partial \xi^l}{\partial x^l} \quad (i, k = 1, \dots, n). \quad (8)$$

Let us extend the operator (7), using the equations (8), to the derivatives $h_{,j}^{ik}$ in accordance with the formulas [138]

$$\begin{aligned} \tilde{X} &= X + \zeta_j^{ik} \frac{\partial}{\partial h_{,j}^{ik}}, \\ \zeta_j^{ik} &= \frac{\partial \eta^{ik}}{\partial x^j} + h_{,j}^{mn} \frac{\partial \eta^{ik}}{\partial h^{mn}} - h_{,l}^{ik} \frac{\partial \xi^l}{\partial x^j} \quad (i, k, j = 1, \dots, n). \end{aligned} \quad (9)$$

Consider the invariance conditions for equations (5):

$$\tilde{X} h_{,k}^{ik} |_{h_{,k}^{jk}=0} = \zeta_k^{ik} |_{h_{,k}^{jk}=0} = 0 \quad (i = 1, \dots, n).$$

Substituting the expressions (9) for ζ_k^{ik} and taking into account equations (5), we reduce these conditions to the form

$$h^{kj} \frac{\partial^2 \xi^i}{\partial x^k \partial x^j} = 0 \quad (i = 1, \dots, n). \quad (10)$$

The equations (10) must hold identically in the variables x^i ($i = 1, \dots, n$) and h^{ik} ($i, k = 1, \dots, n$). Therefore, taking into account the symmetry of the quantities h^{ik} , we have from (10)

$$\frac{\partial^2 \xi^i}{\partial x^k \partial x^j} = 0 \quad (i, k, j = 1, \dots, n).$$

This means that the quantities ξ^i ($i = 1, \dots, n$) are linear functions of x^k ($k = 1, \dots, n$), which proves the lemma.

Lemma 3.2. In order for the group H of transformations (2) to map the set Γ into itself, it is necessary and sufficient that the group \overline{H} be admissible by (1) in the sense of S. Lie.

Proof. *a) Sufficiency.* If the group \overline{H} is admissible by (1), then it follows from general theory [138] that the transformations (2)-(3) map any solution of equations (1) into some solution of (1). In particular, a solution determining the space V_n under consideration is also mapped by the group \overline{H} into a solution of (1). Since the transformations of the group \overline{H} preserve V_n , it follows that the group H maps Γ into itself.

b) Necessity. It is obvious that the linear transformations of coordinates leave the set Γ invariant. Therefore, according to the theorem on the equivalence of quadratic forms of the same signature, one can assign any value to the components of the metric tensor at an arbitrary point $x_0 \in V_n$ by means of an appropriate linear transformation; the transformation leaves the set Γ invariant. Consequently, taking various elements of the set Γ , one can obtain an arbitrary point of the space $E(x, g)$.

Let G be a continuous group of transformations of the space $E(y)$ of variables y^1, \dots, y^N into itself formed by one-parameter continuous Lie groups G_1 . Consider a manifold $\mathcal{M} \subset E(y)$ given by equations

$$\psi^\sigma(y) = 0 \quad (\sigma = 1, \dots, s),$$

and a manifold $\Phi \subset \mathcal{M}$. If any transformation $T \in G$ maps an arbitrary point $y \in \Phi$ into a point $Ty \in \mathcal{M}$ then for all generators X of the group G the following equations hold:

$$X\phi^\sigma(y)|_{y \in \Phi} = 0 \quad (\sigma = 1, \dots, s). \quad (11)$$

Indeed, the operator X generates a one-parameter continuous local Lie group $G_1 \subset G$ with a parameter a (chosen in such a way that $a = 0$ corresponds to the identical transformation). Then, by assumption we have

$$\phi^\sigma(T_a y) = 0 \quad \text{for all } y \in \Phi \quad \text{and } T_a \in G_1.$$

Therefore the well-known formula [138]

$$X\phi^\sigma(y) = \frac{\partial}{\partial a} \phi^\sigma(T_a y)|_{a=0} \quad (\sigma = 1, \dots, s)$$

yields (11).

Let us apply this result to the case when the group G is the extension of \overline{H} to the derivatives $h_{.j}^{ik}$, whereas $E(y)$, \mathcal{M} and Φ are the extended

space $E(x, h, h_{,j}^{ik})$, the manifold given by equations (5) and its submanifold corresponding to the set Γ , respectively. Then equations (11) yield that

$$\widetilde{X}h_{,k}^{ik}|_{h_{,k}^{jk}=0} = 0 \quad (i = 1, \dots, n) \quad (12)$$

for $\{x\} \in \Gamma$.

According to the aforementioned property of Γ to pass through any point of the space $E(x, g)$ or, equivalently, of the space $E(x, h)$, equations (12) should be satisfied identically in the variables x^i ($i = 1, \dots, n$) and h^{ik} ($i, k = 1, \dots, n$). But only these variables appear in equations (12). Therefore, (12) imply equations (10) which provide the necessary and sufficient condition for the invariance of equations (5). This completes the proof of Lemma 3.1, and hence of Theorem 3.1.

Above we confined ourselves to the discussion of groups of transformations of the form (2), (3). One can, however, consider the problem of seeking the broadest continuous group G of transformations of the form:

$$\begin{aligned} x'^i &= \varphi^i(x, g), \\ g'^{ik} &= \psi^{ik}(x, g) \quad (i, k = 1, \dots, n), \end{aligned} \quad (13)$$

composed by one-parameter continuous local Lie groups admitted by equations (1). Since equations (1) reduce to the linear homogeneous equations (5), the group G admitted by these equations contains arbitrary functions. Namely, G contains the subgroup G_0 of the transformations

$$g'^{ik} = a|\det \|\sqrt{|g|}g^{mn} + \varphi^{mn}\|^{1/(2-n)}(\sqrt{|g|}g^{ik} + \varphi^{ik}(x)) \quad (i, k = 1, \dots, n)$$

where $\varphi^{ik}(x)$ is an arbitrary solution of equations (5), and a is a parameter. The subgroup G_0 is a normal divisor of the group G . Hence one can study the factor group G/G_0 and prove the following statement.

Theorem 3.2. The factor group G/G_0 of the broadest continuous group G admitted by equations (1) has the order $n(n+2)$ and consists of the transformations

$$\begin{aligned} x'^i &= \frac{a_k^i x^k + b^i}{c_k x^k + 1}, \\ g'^{ik} &= \left[J \left(\frac{\partial x'}{\partial x} \right) \right]^{\frac{4}{(n+1)(n-2)}} g^{mn} \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n}, \end{aligned} \quad (14)$$

where a_k^i , b^i , c_k are arbitrary constants and $i, k = 1, \dots, n$.

In order to show that transformations (14) are in fact admissible by (1), it is convenient to use equations (1) in the form (5). For the transformations (14), the expressions (8) in the operator (7) are replaced by

$$\eta^{ik} = h^{il} \frac{\partial \xi^k}{\partial x^l} + h^{kl} \frac{\partial \xi^i}{\partial x^l} - \frac{n+3}{n+1} h^{ik} \frac{\partial \xi^l}{\partial x^l} \quad (i, k = 1, \dots, n).$$

Substituting them in (9) and writing down the invariance conditions for (5), one obtains the equations

$$\frac{\partial^2 \xi^i}{\partial x^k \partial x^j} - \frac{1}{n+1} \delta_k^i \frac{\partial^2 \xi^l}{\partial x^j \partial x^l} - \frac{1}{n+1} \delta_j^i \frac{\partial^2 \xi^l}{\partial x^k \partial x^l} = 0 \quad (i, k, j = 1, \dots, n).$$

These equations have the general solution

$$\xi^i = -c_k x^k x^i + a_k^i x^k + b^i \quad (i = 1, \dots, n)$$

in accordance with (14).

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* *Authors note to this 2006 edition:* When this paper was published, I received a letter from V.A. Fock. He asked me if the method might be used for solving the problem on harmonic coordinates in his formulation. See also his remark [49] on Frankle's paper [52].

Paper 4

Group classification of second order differential equations

N.H. IBRAGIMOV [71]

Doklady Akademii Nauk SSSR, Tom 183, (1968), No. 2, pp. 274–277.

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In this paper, based on works [137], [138], the problem of the group classification of second-order differential equations with $n \geq 3$ independent variables is considered. The determining equations of infinitesimal transformations admitted by a certain type of quasi-linear second-order equations of a physical interest are derived. An invariant Laplace equation in a Riemannian space is introduced.

1. In physics (see [98] and the bibliography therein) equations of the following type are considered:

$$F[u] \equiv g^{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i(x) \frac{\partial u}{\partial x^i} + \psi(x, u) = 0, \quad (1)$$

$$(g^{ij} = g^{ji}; \quad i, j = 1, \dots, n; \quad \det \|g^{ij}\| \neq 0),$$

where $n = 4$, and the terms with derivatives provide simply the wave operator. We will consider an arbitrary equation (1) with $n \geq 3$.

Let us introduce the Riemannian space V_n with the metric tensor $g_{ij}(x)$ determined by the equations $g_{ik}g^{kj} = \delta_i^j$ ($i, j = 1, \dots, n$). In what follows, the usual tensor notation is utilized. Namely, Γ_{ij}^k are the Christoffel symbols, R_{ij} is the Ricci tensor and $R = R_i^i \equiv g^{ij}R_{ij}$ is the scalar curvature of the space V_n . Indices after the comma will denote the covariant differentiation in

V_n . All equations will be considered up to equivalence, where the following transformations are called equivalence transformations:

$$x'^i = x'^i(x) \quad (i = 1, \dots, n), \quad (2)$$

$$F'[u] = e^{-f(x)} F[ue^{f(x)}], \quad (3)$$

$$F'[u] = \Phi(x) F[u]. \quad (4)$$

We consider the problem of determining the continuous groups of transformations admitted by equations (1) in the sense of S. Lie. Omitting standard calculations, I present here only the results.

The infinitesimal operator of the continuous one-parameter local Lie group admitted by equation (1) has the form

$$X = \xi^i(x) \frac{\partial}{\partial x^i} + [\sigma(x)u + \tau(x)] \frac{\partial}{\partial u} \quad (5)$$

and is determined by the equations

$$\begin{aligned} \xi_{i,j} + \xi_{j,i} &= \mu(x) g_{ij}, \\ 2\sigma_{,i} &= \frac{2-n}{2} \mu_{,i} - (a_j \xi^j)_{,i} - K_{ij} \xi^j, \\ (K_{il} \xi^l)_{,j} &= (K_{jl} \xi^l)_{,i}, \\ (\xi^k H_{,k} + \mu H)u + X(\psi) + (\mu - \sigma)\psi + g^{ik} \frac{\partial^2 \tau}{\partial x^i \partial x^k} + b^i \frac{\partial \tau}{\partial x^i} &= 0, \end{aligned} \quad (6)$$

where

$$\begin{aligned} a^i &= b^i + g^{jk} \Gamma_{jk}^i, \\ K_{ij} &= a_{i,j} - a_{j,i}, \\ H &= -\frac{1}{2} \left(a^i_{,i} + \frac{1}{2} a^i a_i + \frac{n-2}{2(n-1)} R \right). \end{aligned}$$

It follows from the first line of equations (6) that, as in the case of linear equations [137], the quantities ξ^i define a subgroup of the group of conformal transformations of the space V_n . It is known that the maximal number of parameters of the group of conformal transformations of the space V_n ($n \geq 3$) does not exceed $(n+1)(n+2)/2$, and this number is reached only for conformally flat spaces [169]. For groups of motions, the maximal number of parameters is equal to $n(n+1)/2$, and this is reached only for spaces of constant curvature [46]. Using these facts, we arrive, after some calculations, at the following theorem.

Theorem 4.1. Equations of type (1) which admit the group of motions of the space V_n of maximal order $n(n+1)/2$ are equivalent, with respect to the transformations (2), (3), (4) to one of the following equations:

$$\Delta u + u^{(n+2)/(n-2)} \varphi(u(1+r^2)^{(n-2)/2}) = 0 \quad \left(r^2 = \sum_{i=1}^n e^i (x^i)^2 \right) \quad (7)$$

in the case of nonzero constant curvature of the space V_n , or

$$\Delta u + \varphi(u) = 0 \quad (8)$$

in the case of zero curvature, where φ is an arbitrary function. If equation (1) admits the group of conformal transformations of the space V_n of maximal order $(n+1)(n+2)/2$ it is equivalent to the equation

$$\Delta u + au^{(n+2)/(n-2)} = 0, \quad (9)$$

where a is an arbitrary constant.

In Theorem 4.1, Δ is the operator

$$\Delta = \sum_{i=1}^n e_i \frac{\partial^2}{(\partial x^i)^2}, \quad e_i = \pm 1,$$

where the signature of the operator coincides with the signature of g^{ij} . Henceforth we shall call Δ a Laplace operator, ignoring the signature.

Corollary 4.1. Every linear differential equation of second order, invariant with respect to a $((n+1)(n+2)/2)$ -parameter continuous group of transformations (disregarding the transformations $u' = au + \phi(x)$, where $a = \text{const.}$ and $\phi(x)$ is an arbitrary solution of the equation in question) reduces, by means of the transformations (2) – (4), to the Laplace equation

$$\Delta u = 0. \quad (10)$$

2. The above discussion manifests that, in considering linear second-order equations, the Laplace equation (10) is characteristic for all conformally flat Riemannian spaces. Let us now introduce an invariant Laplace equation (in particular, a wave equation in the case of the corresponding signature) in a Riemannian space possessing the similar property with respect to mutually conformal spaces. It is generally accepted to consider as a Laplace equation in a Riemannian space the equation

$$\Delta_2 u = g^{ij} u_{,ij} = 0, \quad (11)$$

obtained merely by replacing the usual partial derivatives by covariant ones. A basis for this is provided by the external resemblance between equation (11) and the usual Laplace equation (10), and also by the fact that (11) turns into (10) when the space V_n is flat. There is, however, an essential difference between equations (10) and (11), namely, that if (10) is invariant with respect to the group of conformal transformations of a flat space, equation (11) is invariant, in general, only with respect to the group of motions, and not to conformal transformations of the corresponding space V_n (i.e. of the space in which the components of the metric tensor are the coefficients g^{ij} of the second derivatives). This is easy to check, for example, when equation (11) is written in a space of constant nonzero curvature, by solving the determining equations (6). It seems reasonable, if we start from the properties of the usual Laplace equation (10), to require that the Laplace equation in V_n satisfies the following conditions:

- 1°. Linearity and homogeneity.
- 2°. General covariance.
- 3°. Invariance with respect to the group of conformal transformations of the space V_n .

Theorem 4.2. In any Riemannian space V_n , the equation

$$\Delta u = g^{ij}u_{,ij} + \frac{n-2}{4(n-1)}Ru = 0 \quad (12)$$

satisfies conditions 1°, 2° and 3°. For a conformally flat space such an equation is unique to within the equivalence transformation (3).

On the basis of this theorem the following definition is suggested.

Definition 4.1. Equation (12) is called an invariant Laplace equation in a Riemannian space V_n , and the operator Δ appearing in (12) is called an invariant Laplace operator in V_n .

According to Definition 4.1, the invariant Laplace equation in the spaces V_n with zero scalar curvature R has the form (11). This occurs, for example, for the Einstein spaces ($R_{ij} = 0$, $i, j = 1, \dots, n$).

The invariant Laplace equation has the following remarkable property.

Theorem 4.3. Let V_n and \bar{V}_n be conformal Riemannian spaces with respective metric tensors

$$g_{ij}(x), \quad \bar{g}_{ij}(x) = e^{2\theta(x)}g_{ij}(x).$$

Then the invariant Laplace equations in these spaces are mutually equivalent. The equivalence transformation has the form

$$\bar{\Delta}u = e^{-\frac{n+2}{2}\theta} \Delta(ue^{\frac{n-2}{2}\theta}), \quad (13)$$

where Δ and $\bar{\Delta}$ are the invariant Laplace operators (12) for the spaces V_n and \bar{V}_n , respectively.

Equations (7) and (8) can be combined and written in the form (8) by assuming that Δ denotes the invariant Laplace operator in a space of constant curvature. Then the standard forms (found in [137]) of linear second-order equations admitting groups of motions of maximal order assume the form

$$\Delta u - \lambda u = 0 \quad (\lambda = \text{const.}) \quad (14)$$

with an invariant Laplace operator Δ . It follows from [137] that in a space of constant curvature, equation (14) is a unique (to within equivalence transformations) covariant equation admitting a group of motions of maximal order. Equation (14) written for an arbitrary Riemannian space V_n will be called an *invariant Klein-Gordon equation in V_n* .

As an example, let us discuss an application of the invariant Laplace equation introduced here for an exposition of the red shift theory based on the hypothesis that space-time is a Friedmann-Lobachevsky space [50]. Let us take the square of the interval of the Friedmann-Lobachevsky space in the form

$$ds^2 = H^2(r)(dt^2 - dx^2 - dy^2 - dz^2) \quad (15)$$

without specifying the form of the function $H(r)$. Let us assume that the propagation of the light in V_4 is governed by a wave equation in this space. If we accept the invariant Laplace equation (12) as the wave equation in V_n , then the wave equation

$$\Delta\psi = 0 \quad (16)$$

in the Friedmann-Lobachevsky space is written, by virtue of equation (13), in the following form:

$$\Delta\psi = \frac{1}{H^3} \square(\psi H), \quad (17)$$

where \square is the usual wave operator in the flat space. Setting

$$\psi^* = \psi H,$$

we obtain from (16) and (17) the equation

$$\square\psi^* = 0. \quad (18)$$

On the other hand, if one takes for the wave equation in a Riemannian space the equation (11), then, instead of (18), one obtains ([50], § 95)

$$H\Box\psi^* - \psi^*\Box H = 0, \quad (19)$$

from which, using the special form of the function $H(r)$ and ignoring the term $\Box H$, one approximately obtains equation (18). The consequences for physics of the transition from a wave equation in a Friedmann-Lobachevsky space to the usual wave equation (18) are discussed in [50].

It is interesting to note the following. The simplification of equation (19) carried out in [50] shows that the function ψ in fact satisfies not equation (11), which it should satisfy, but equation (12). This leads to equation (12) by a different (random) method. Here I have introduced the invariant Laplace equation (12) formally, starting only from group theoretic considerations. Equally formal, of course, is the hypothesis in the above example that the transmission of light in the space V_4 is described by an invariant Laplace equation (12). The question of the physical content of the invariant Laplace equation in V_n (and in particular, the question of the correspondence between equation (12) and the Maxwell equations in V_4) remains open.

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Paper 5

Invariance of Dirac's equations

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The article deals with the problem of invariance of Dirac's equations with respect to a continuous group of transformations of Pauli type. The most general groups of transformations of this type admissible in Lie's sense by Dirac's equations are found for nonzero and zero mass. In the case of zero mass the group has 7 essential parameters and contains the well-known 4-parameter Pauli group as a subgroup. A 6-parameter subgroup of this group is determined, which is isomorphic to the group of 4-dimensional rotations. In the case of nonzero mass this group turns out to be a 3-parameter group isomorphic to the group of three-dimensional rotations.

Pauli discovered [146] a three-parameter continuous group of transformations with respect to which the Dirac's equations

$$(\gamma_\mu \frac{\partial}{\partial x^\mu} + m)\psi = 0 \quad (1)$$

are invariant in case $m = 0$, by considering the transition to the complex conjugate function of ψ . In this case, besides equations (1), the conjugate equations must also be taken into account. The latter can be written in the form

$$(\gamma_\mu^T \frac{\partial}{\partial x^\mu} - m)\bar{\psi} = 0 \quad (2)$$

with the function

$$\bar{\psi} = \gamma_4 \psi^*. \quad (3)$$

The star and T denote complex conjugate and transposition, respectively. This group, together with the transformation group

$$\begin{aligned} \psi' &= (\cos \alpha + i \sin \alpha \cdot \gamma_5) \psi, \\ \bar{\psi}' &= (\cos \alpha + i \sin \alpha \cdot \gamma_5) \bar{\psi}, \end{aligned} \quad (4)$$

where $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$, α is a real parameter, is called the 4-parameter Pauli group. The structure of the Pauli group is studied in [132].

We will show that the Pauli group is not the most general continuous group of transformations

$$\begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix} = A \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad (5)$$

where $\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$ is an 8-dimensional column vector, A is a complex 8 by 8 matrix (in general depending upon x), admissible in Lie's sense by equations (1) and (2). Furthermore, we will find the most general group both for case $m = 0$ and m arbitrary. Observe that considering the pair of complex equations (1) and (2) is equivalent to dividing equation (1) into real and imaginary parts. In the latter case it is natural to consider the group of real transformations. The condition that the transformation is real leads in this case to the equality

$$\bar{\psi}' = \overline{\psi'}. \quad (6)$$

From (5) and (6) it follows that the matrix A has the form (γ_4 is real)

$$A = \begin{pmatrix} a & \vdots & b \\ \dots & \vdots & \dots \\ \gamma_4 b^* \gamma_4 & \vdots & \gamma_4 a^* \gamma_4 \end{pmatrix} \quad (7)$$

with arbitrary 4×4 matrices a and b . The equivalence of conditions (6) and (7) is easily verified.

To describe explicitly an admissible group we choose a certain representation of matrices γ_μ ($\mu = 1, \dots, 4$), namely:

$$\begin{aligned}
\gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \gamma_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \tag{8}$$

The following properties of matrices (8) will be useful:

$$\gamma_1^T = -\gamma_1, \quad \gamma_2^T = \gamma_2, \quad \gamma_3^T = -\gamma_3, \quad \gamma_4^T = \gamma_4, \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}. \tag{9}$$

By the well-known method in the theory of group properties of differential equations [138] we can find most general admissible groups. The results obtained will be presented in two theorems valid for case $m = 0$ and $m \neq 0$ respectively.

Theorem 5.1. The most general continuous group of transformations (5) satisfying condition (6), which is Lie-admissible by equations (1), (2) for $m = 0$, contains 7 essential parameters. It consists of a 6-parameter group G_6 isomorphic with the group of 4-dimensional rotations O_4 and of the group of transformations (4). The 3-parametric Pauli group is a subgroup of the group G_6 .

Next we shall describe the basis of infinitesimal operators (generators) of the group G_6 and determine an isomorphism between G_6 and O_4 . Using matrices (8), the basis operators of the group G_6 may be written in the following form of 8×8 matrices:

$$\begin{aligned}
S_1 &= \begin{pmatrix} 0 & \vdots & \gamma_4 \gamma_2 \\ \dots & \vdots & \dots \\ -\gamma_4 \gamma_2 & \vdots & 0 \end{pmatrix}, & S_2 &= \begin{pmatrix} 0 & \vdots & i\gamma_4 \gamma_2 \\ \dots & \vdots & \dots \\ i\gamma_4 \gamma_2 & \vdots & 0 \end{pmatrix}, \\
S_3 &= \begin{pmatrix} 0 & \vdots & \gamma_3 \gamma_1 \\ \dots & \vdots & \dots \\ \gamma_3 \gamma_1 & \vdots & 0 \end{pmatrix}, & S_4 &= \begin{pmatrix} 0 & \vdots & i\gamma_3 \gamma_1 \\ \dots & \vdots & \dots \\ -i\gamma_3 \gamma_1 & \vdots & 0 \end{pmatrix},
\end{aligned} \tag{10}$$

$$S_5 = \begin{pmatrix} -\gamma_5 & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & \gamma_5 \end{pmatrix}, \quad S_6 = \begin{pmatrix} -i1 & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & i1 \end{pmatrix}.$$

Here 0 and 1 denote the zero and identity matrices, respectively. Transformations of group G_6 can be written easily in view of the equations

$$S_1^2 = S_2^2 = S_5^2 = 1, \quad S_3^2 = S_4^2 = S_6^2 = -1, \quad (11)$$

making use of the formula*

$$\begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix} = e^{\alpha S} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad (12)$$

where α is a real parameter.

As an example we shall write down one-parameter groups of transformations for operators S_1 and S_3 . In view of (11) we have:

$$e^{\alpha S_1} = 1 + \alpha S_1 + \frac{\alpha^2}{2!} S_1^2 + \dots = \cosh \alpha + S_1 \sinh \alpha.$$

Thus formula (12) determines a one-parameter group

$$\psi' = \cosh \alpha \cdot \psi + \sinh \alpha \cdot \gamma_4 \gamma_2 \bar{\psi}, \quad \bar{\psi}' = \cosh \alpha \cdot \bar{\psi} - \sinh \alpha \cdot \gamma_4 \gamma_2 \psi. \quad (13)$$

For S_3 one has similarly

$$\begin{aligned} e^{\alpha S_3} &= \cos \alpha + S_3 \sin \alpha, \quad \psi' = \cos \alpha \cdot \psi + \sin \alpha \cdot \gamma_3 \gamma_1 \bar{\psi}, \\ \bar{\psi}' &= \cos \alpha \cdot \bar{\psi} + \sin \alpha \cdot \gamma_3 \gamma_1 \psi. \end{aligned} \quad (14)$$

Comparing the transformations of group G_6 with the Pauli group, it is possible to make the following statement. Operators S_3, S_4 and S_6 represent the basis operators of the 3-parameter Pauli group in representation (8). One-parameter groups generated by operators S_1, S_2, S_5 are not contained in the Pauli group. So far as operators S_1 and S_2 are concerned the latter fact can be obtained using a different reasoning (it is done in Remark 5.2).

We shall construct the isomorphism of groups G_6 and O_4 . Let us denote

$$\begin{aligned} X_{12} &= \frac{i}{2} S_2, & X_{23} &= \frac{1}{2} S_4, & X_{31} &= -\frac{i}{2} S_5, \\ X_{14} &= \frac{i}{2} S_1, & X_{24} &= -\frac{1}{2} S_6, & X_{34} &= -\frac{1}{2} S_3. \end{aligned} \quad (15)$$

*The connection between the expression of infinitesimal generators in Lie's form and the matrix form (10) is given in [70]. Formula (12) is equivalent to finding the transformations of the group by solving Lie's equations.

It follows from [70] that to commutators of infinitesimal operators in Lie's form there correspond "commutators" of matrices (10) defined by

$$[S_i, S_j] = S_j S_i - S_i S_j. \quad (16)$$

Calculating the commutators of operators (15) by means of formulas (16) we arrive at the structure of the group O_4 :

$$[X_{\mu\nu}, X_{\sigma\tau}] = \delta_{\mu\sigma} X_{\nu\tau} + \delta_{\nu\tau} X_{\mu\sigma} - \delta_{\mu\tau} X_{\nu\sigma} - \delta_{\nu\sigma} X_{\mu\tau}. \quad (17)$$

Furthermore, the infinitesimal operator of the group (4) commutes with all operators (10).

Theorem 5.2. The most general continuous group of transformations (5) under the condition (6) admitted by equations (1), (2) for $m \neq 0$ has 3 parameters and is isomorphic to the group of three-dimensional rotations O_3 . As basis operators of this group one can choose S_1, S_2 and S_6 .

In both Theorem 5.1 and Theorem 5.2 a trivial transformation, namely the dilation of functions ψ and $\bar{\psi}$ was not taken into account.

The invariance of Dirac's equations (1), (2) with respect to one-parameter transformation groups given in Theorem 5.2 can be easily verified. We shall do this e.g. for the operator S_1 . The one-parameter transformation group for this operator has the form (13). Substituting (13) into (1), (2) and using (9) one obtains

$$\begin{aligned} & (\gamma_\mu \frac{\partial}{\partial x_\mu} + m) \psi' \\ &= \cosh \alpha (\gamma_\mu \frac{\partial}{\partial x^\mu} + m) \psi + \sinh \alpha (\gamma_\mu \frac{\partial}{\partial x^\mu} + m) \gamma_4 \gamma_2 \bar{\psi} \\ &= \sinh \alpha \cdot \gamma_4 \gamma_2 (\gamma_1 \frac{\partial}{\partial x^1} - \gamma_2 \frac{\partial}{\partial x^2} + \gamma_3 \frac{\partial}{\partial x^3} - \gamma_4 \frac{\partial}{\partial x^4} + m) \bar{\psi} \\ &= \sinh \alpha \cdot \gamma_4 \gamma_2 (-\gamma_\mu^T \frac{\partial}{\partial x^\mu} + m) \bar{\psi} = 0, \end{aligned}$$

and

$$\begin{aligned} & (\gamma_\mu^T \frac{\partial}{\partial x^\mu} - m) \bar{\psi}' \\ &= \cosh \alpha (\gamma_\mu^T \frac{\partial}{\partial x^\mu} - m) \bar{\psi} - \sinh \alpha (\gamma_\mu^T \frac{\partial}{\partial x^\mu} - m) \gamma_4 \gamma_2 \psi \\ &= \sinh \alpha \cdot \gamma_4 \gamma_2 (\gamma_\mu \frac{\partial}{\partial x^\mu} + m) \psi = 0. \end{aligned}$$

These equations prove the invariance of Dirac's equations.

Remark 5.1. The condition $m = 0$ follows from the invariance of Dirac's equations (1), (2) with respect to any one-parameter group with an operator which is a linear combination of the operators S_3 , S_4 , S_5 , and also with respect to the group (4).

Remark 5.2. No linear combination of the operators S_1 and S_2 can belong to the operators of the Pauli group. In fact, if this were possible at least for one operator obtained in such a way, we should obtain a one-parameter group belonging to the Pauli group with respect to which the Dirac's equations with mass different from zero are invariant, which is impossible ([132], p. 368 in Russian translation).

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Paper 6

Groups of generalized motions

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This article is devoted to a study of groups of generalized motions in Riemannian spaces [68], [69]. An analogue of Killing's equations is obtained for arbitrary groups of generalized motions. A connection is established between the set of all spaces conformal to a given Riemannian space and the conformal group in this space. All the statements in this article have local character.

§ 1 Defect of invariance

The basic working idea of this article is that of the defect of invariance of a manifold with respect to a continuous group of transformations, introduced by L. V. Ovsyannikov [140], [138]. We therefore discuss below the basic facts connected with the defect of invariance.

We denote by H a continuous group of transformations (we shall simply say the group H) of an n -dimensional Euclidean space $E(x)$ of points $x = (x^1, \dots, x^n)$ into itself. An element of H corresponding to a group parameter $a = (a^1, \dots, a^r)$ will be denoted by T_a , and the transformation will be written in the form

$$x'^i = x'^i(x, a) \quad (i = 1, \dots, n). \quad (1)$$

We write the basic infinitesimal operators of H in the form

$$X_\alpha = \xi_\alpha^i(x) \frac{\partial}{\partial x^i} \quad (\alpha = 1, \dots, r). \quad (2)$$

Consider a manifold $\mathcal{N} \subset E(x)$ of dimension $\dim \mathcal{N} = n - s$, regularly defined [140] by a system of equations

$$\psi^\sigma(x) = 0 \quad (\sigma = 1, \dots, s), \quad (3)$$

and introduce the notation

$$T_a(\mathcal{N}) = \bigcup_{x \in \mathcal{N}} T_a x, \quad H(\mathcal{N}) = \bigcup_{T_a \in H} T_a(\mathcal{N}). \quad (4)$$

If $\mathcal{M} \subset E(x)$ is some manifold which contains \mathcal{N} and has the property $H(\mathcal{M}) = \mathcal{M}$ (in particular, $H(\mathcal{N})$ is such a manifold), then

$$H(\mathcal{N}) \subset \mathcal{M}. \quad (5)$$

Definition 6.1. The nonnegative integer

$$\delta(\mathcal{N}, H) = \dim H(\mathcal{N}) - \dim \mathcal{N} \quad (6)$$

is called the defect of invariance of \mathcal{N} with respect to H .

We denote by $R(\|M(x)\|_{\mathcal{N}})$ the general rank of the matrix $\|M(x)\|$ evaluated on \mathcal{N} . A convenient method of calculating the defect of invariance gives the following important theorem [140].

Theorem 6.1.

$$\delta(\mathcal{N}, H) = R(\|X_\alpha \psi^\sigma(x)\|_{\mathcal{N}}). \quad (7)$$

§ 2 Riemannian space

It is convenient for us to take the following point of view of an n -dimensional Riemannian space V_n . Consider a Euclidean space $E(x, g)$ of variables x^i ($i = 1, \dots, n$), g_{ij} ($i, j = 1, \dots, n$) and all possible n -dimensional manifolds $\bar{V}_n \subset E(x, g)$ defined by the equations

$$g_{ij} = \varphi_{ij}(x), \quad (\det \|\varphi_{ij}\| \neq 0; \quad \varphi_{ij} = \varphi_{ji}; \quad i, j = 1, \dots, n). \quad (8)$$

We shall say that a manifold \bar{V}'_n defined by the equations $g_{ij} = \varphi'_{ij}(x)$ is equivalent to \bar{V}_n and write $\bar{V}'_n \sim \bar{V}_n$, if there exist functions $x'^i = x'^i(x)$ ($i = 1, \dots, n$) satisfying the equations

$$\varphi'_{kl}(x'(x)) \frac{\partial x'^k(x)}{\partial x^i} \frac{\partial x'^l(x)}{\partial x^j} = \varphi_{ij}(x) \quad (i, j = 1, \dots, n). \quad (9)$$

Then the class of all mutually equivalent manifolds \bar{V}_n will define a Riemannian space V_n . We shall say that the manifold \bar{V}_n defines a space V_n in the system of coordinates x^i , and the manifold $\bar{V}'_n \sim \bar{V}_n$ in the system of coordinates x'^i ($i = 1, \dots, n$). This method of defining a Riemannian space agrees with the usual one [46].

§ 3 Generalized motions

Suppose we are given a group H of transformations (1), and suppose the quantities g_{ij} ($i, j = 1, \dots, n$) are transformed by it according to the formulae,

$$g'_{ij} = g_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \quad (i, j = 1, \dots, n). \quad (10)$$

Then to the group H there corresponds a group \bar{H} of transformations (1), (10) of the space $E(x, g)$ into itself. The expression for the infinitesimal operators of \bar{H} is given in [68], [69]. Consider a manifold \bar{V}_n defining a space V_n in some coordinate system and give the following definition.

Definition 6.2. The number

$$\delta(V_n, H) = \delta(\bar{V}_n, \bar{H}) \quad (11)$$

is called *the defect of the Riemannian space V_n with respect to the group H* , and H is called *the group of generalized motions of defect $\delta(V_n, H)$ in the Riemannian space V_n* .

We denote by $\|\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i}\|$ the matrix whose rows are numbered by the index α and the columns by the double index ij . For convenience, the index α is written in brackets, and the indices after the comma denote covariant differentiation in V_n . The following theorem gives an analogue of the Killing equations for groups of generalized motions.

Theorem 6.2.

$$\delta(V_n, H) = R(\|\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i}\|). \quad (12)$$

Proof. Formula (12) follows from Theorem 6.1 applied to the manifold \bar{V}_n and the group \bar{H} .

Corollary 6.1. In order that H should be a group of motions in the Riemannian space V_n , it is necessary and sufficient that the equation

$$\delta(V_n, H) = 0$$

should be satisfied.

On the right-hand side of equation (11), as well as of (12), a concrete representative \bar{V}_n of the space V_n appears explicitly. In order that Definition 6.2 should be correct, we must show that the number $\delta(V_n, H)$ does not depend on the choice of coordinate system in V_n , i.e. on the choice of \bar{V}_n .

Theorem 6.3. The defect $\delta(V_n, H)$ does not depend on the choice of coordinate system in V_n .

Proof. Under a transformation of coordinates in V_n the columns of the matrix

$$\|\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i}\|$$

are transformed like the components of a second-order tensor. Therefore the columns of the transformed matrix are linear combinations of the columns of the original matrix, and so the rank of the matrix does not change. By virtue of (12) this means that the theorem is true.

§ 4 The invariant class of spaces and the invariants of a space

Let us clarify the meaning of the manifold $\bar{T}_a(\bar{V}_n)$ and $\bar{H}(\bar{V}_n)$ constructed according to formulae (4). The manifold $\bar{T}_a(\bar{V}_n)$ is defined by the equations

$$g_{ij} = \varphi_{kl}(x'(x, a)) \frac{\partial x'^k(x, a)}{\partial x^i} \frac{\partial x'^l(x, a)}{\partial x^j} \quad (i, j = 1, \dots, n). \quad (13)$$

Therefore

$$\bar{T}_a(\bar{V}_n) \sim \bar{V}_n \quad (14)$$

for all $T_a \in H$.

It follows from the equation (see Definitions 6.1 and 6.2)

$$\dim \bar{H}(\bar{V}_n) = n + \delta(V_n, H) \quad (15)$$

and Theorem 3 that the manifold $\bar{H}(\bar{V}_n)$ defines, in a definite coordinate system, a certain class of Riemannian spaces depending on $\delta(V_n, H)$ arbitrary functions. Let us denote this class of spaces by $H(V_n)$.

Definition 6.3. The class of Riemannian spaces $H(V_n)$ is called *the invariant class of spaces* corresponding to the pair (V_n, H) .

The construction of the invariant class $H(V_n)$, starting from a manifold $\bar{H}(\bar{V}_n)$, is carried out according to § 2. The basic property of the invariant class $H(V_n)$ is that it is the smallest set of spaces which contains V_n and is invariant with respect to all transformations of the group H . This follows from (5).

It follows from Definition 6.3 that there exist exactly

$$\frac{n(n+1)}{2} - \delta(V_n, H)$$

distinctly different geometric quantities which have the same values in all spaces belonging to the class $H(V_n)$ at the points with the same values of the coordinates x^i ($i = 1, \dots, n$). In particular, taking the space V_n and invoking (14), we see that there exist $\frac{n(n+1)}{2} - \delta(V_n, H)$ independent geometric elements in V_n that are invariant with respect to the group H . I call them *invariants* of the space V_n with respect to the group H .

§ 5 Conformal group and conformal spaces

Let us consider, as a group of generalized motions, the group of conformal transformations in V_n . In this case we can establish an interesting connection between the conformal group and conformal spaces (Theorem 6.6). Usually they are considered independently.

A space $V_n^{(\sigma)}$ with metric tensor

$$g_{ij} = \sigma(x)\varphi_{ij}(x) \quad (i, j = 1, \dots, n) \quad (16)$$

is said to be conformal to a space V_n with metric tensor (8). Regarding $\sigma(x)$ as an arbitrary function, one obtains the whole class of spaces conformal to a given space V_n . We denote this class of spaces by $\{V_n^{(\sigma)}\}$. It is defined by the manifold

$$\{\bar{V}_n^{(\sigma)}\} = \bigcup_{\sigma} \bar{V}_n^{(\sigma)}.$$

Definition 6.4. A group H is called the *group of conformal transformations* in a space V_n if

$$\bar{T}_a(\bar{V}_n) \subset \{\bar{V}_n^{(\sigma)}\} \quad (17)$$

for any $T_a \in H$.

In what follows, we consider spaces V_n where the conformal group is wider than the group of motions.

Theorem 6.4. Let H be the group of conformal transformations in V_n . Then

$$\delta(V_n, H) = 1. \quad (18)$$

Proof. The coordinates of the operators (2) of the group H satisfy the equations [46]

$$\xi_{(\alpha)i,j} + \xi_{(\alpha)j,i} = \mu_{\alpha}(x)\varphi_{ij}(x) \quad (i, j = 1, \dots, n; \alpha = 1, \dots, r). \quad (19)$$

Therefore equation (12) yields:

$$\delta(V_n, H) = R(\|\mu_{\alpha}(x)\varphi_{ij}(x)\|) = 1. \quad (20)$$

Theorem 6.5. The group H of conformal transformations in a space V_n is the conformal group in any space $V_n^{(\sigma)}$.

Proof. By virtue of (13), the condition (17) means that

$$\varphi_{kl}(x'(x, a)) \frac{\partial x'^k(x, a)}{\partial x^i} \frac{\partial x'^l(x, a)}{\partial x^j} = f(x, a) \varphi_{ij}(x) \quad (i, j = 1, \dots, n). \quad (21)$$

Multiplying both sides of (21) by the function $\sigma(x'(x, a))$ and setting

$$f_\sigma(x, a) = \frac{\sigma(x'(x, a))}{\sigma(x)} f(x, a)$$

one obtains

$$[\sigma(x'(x, a)) \varphi_{kl}(x'(x, a))] \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^l}{\partial x^j} = f_\sigma(x, a) [\sigma(x) \varphi_{ij}(x)] \quad (i, j = 1, \dots, n). \quad (22)$$

It follows that

$$\bar{T}_a(\bar{V}_n^{(\sigma)}) \subset \{\bar{V}_n^{(\sigma)}\}.$$

This proves the theorem.

Theorem 6.6. Let H be the conformal group in V_n . Then

$$H(V_n) = \{V_n^{(\sigma)}\}. \quad (23)$$

Proof. It follows from Theorem 6.5 that

$$\bar{H}(\{\bar{V}_n^{(\sigma)}\}) = \{\bar{V}_n^{(\sigma)}\}.$$

Therefore, by virtue of (5), we have

$$\bar{H}(\bar{V}_n) \subset \{\bar{V}_n^{(\sigma)}\}.$$

Thus, using the equation

$$\dim\{\bar{V}_n^{(\sigma)}\} = \dim\bar{H}(\bar{V}_n) = n + 1,$$

and taking account the local character of all considerations, we conclude that

$$\bar{H}(\bar{V}_n) = \{\bar{V}_n^{(\sigma)}\}.$$

This means precisely that (23) is satisfied. This proves the theorem.

§ 6 Example

Consider the flat space S_n with the metric tensor given by the equations

$$g_{ij} = \delta_{ij} \quad (i, j = 1, \dots, n),$$

and an infinite group H with the basic infinitesimal generators

$$X_\mu = \frac{\partial}{\partial x^\mu}, \quad X_{\nu\mu} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} \quad (\nu < \mu), \quad \nu, \mu = 1, \dots, n-1;$$

$$X_f = f(x^n) \frac{\partial}{\partial x^n},$$

where $f(x^n)$ is an arbitrary function.

It is not difficult to establish that $\delta(S_n, H) = 1$. The invariant class $H(S_n)$ is defined in a given coordinate system by the equations

$$g_{\mu i} = \delta_{\mu i} \quad (\mu = 1, \dots, n-1; i = 1, \dots, n), \quad g_{nn} = F,$$

where $F = F(x^1, \dots, x^n)$ is an arbitrary function.

Let us find the invariants of S_n with respect to H . To this end, we consider an elementary n -hedron which is defined by giving all its $\frac{n(n+1)}{2}$ edges. The lengths of the $\frac{n(n-1)}{2}$ edges of the n -hedron which lie in the $(n-1)$ -dimensional subspace of the variables (x^1, \dots, x^{n-1}) , and the $n-1$ ratios of the lengths of the remaining n edges, are the same for all the spaces of the class $H(S_n)$. The general number of identical quantities is thus equal to $\frac{n(n+1)}{2} - 1$. These quantities are invariants of S_n with respect to H .

If, instead of an n -hedron, we take an elementary n -dimensional ball in S_n , then under the transformations of H this ball is stretched along the axis x^n , while all the other dimensions are preserved.

It should be noted that, in contrast to the number of invariants which is independent on the choice of coordinate system in the space, the concrete form of these invariants does depend on the choice of coordinate system.

§ 7 Application to Einstein's equations

One of the difficulties in studying Einstein's equations of the general theory of relativity is the large number of unknown functions, namely the ten components of the metric tensor of the space V_4 . In order to simplify these equations we can, for example, confine ourselves to the class of conformally flat spaces. This reduces the number of unknown functions to one, and the solutions obtained have a satisfactory physical interpretation [168].

Theorem 6.6 indicates how one can similarly use the invariant classes of spaces of different pairs (V_4, H) . Namely, if one is looking for a solution of Einstein's equations which belongs to the class $H(V_4)$, then the number of unknown functions is equal to $\delta(V_4, H)$. The problem of simplifying Einstein's equations is thus reduced to the problem of a physically reasonable choice of the original space V_4 and the group of generalized motions in this space with a small defect.

This article owes a great deal to conversations with L. V. Ovsyannikov, which were not limited to discussion of the results obtained. The constructive criticism of A. D. Aleksandrov was also very valuable. The author expresses his sincere thanks to them.

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Paper 7

The wave equation in Riemannian spaces

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Introduction

The problem of determining a wave equation in a Riemannian space is nontrivial. Equation

$$g^{ij}\varphi_{,ij} = 0, \quad (1)$$

where indices after the comma indicate covariant differentiation, is usually considered [42], [50], [168] to be a wave equation in the Riemannian space V_n with the metric tensor g^{ij} ($i, j = 1, \dots, n$) although it does not possess the important property of a regular wave equation, namely the property of conformal invariance*.

It was suggested in [71] to take the equation

$$g^{ij}\varphi_{,ij} + \frac{n-2}{4(n-1)} R \varphi = 0, \quad (2)$$

where R is the scalar curvature of the space V_n , as the wave equation in the space V_n . A remarkable property of equation (2) is that it is the only equation which is invariant with respect to the group of conformal transformations in the space V_n for any Riemannian space V_n . Moreover, utilization

*Probably, this fact was first noticed by Schrödinger and Pauli [145].

of equation (2) as the wave equation, e.g. in the study of the red shift provides the exact formula instead of an approximate result obtained in [50] by means of equation (1) (see also [71]).

The present paper gives a detailed presentation of the results of the paper [71] concerning the wave equation in V_n .

§ 1 Covariant form of the wave equation

First let us check how the wave equation

$$\square\varphi \equiv \frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} = 0 \quad (3)$$

usually leads to equation (1).

In order to write equation (3) in an arbitrary coordinate system let us introduce the values

$$g^{11} = g^{22} = g^{33} = -1, \quad g^{44} = 1, \quad g^{ij} = 0 \quad (1 \neq j; i, j = 1, \dots, 4) \quad (4)$$

and denote

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = t.$$

Then equation (3) can be written in the form

$$g^{ij} \frac{\partial^2\varphi}{\partial x^i \partial x^j} = 0. \quad (5)$$

Equations (4) define components of the metric tensor for the 4-dimensional flat Riemannian space V_4 , i.e. the Minkowsky space. In a coordinate system where the metric tensor has the form (4), the covariant differentiation in V_4 is identical with the usual differentiation. Therefore, equation (5) can be also written in the form

$$g^{ij} \varphi_{,ij} = 0, \quad (6)$$

which does not depend on the choice of a coordinate system. Equation (6) is the covariant form of the wave equation (5).

It is obvious that the generalization of equation (6) to an n -dimensional flat Riemannian space V_n is obtained by extending the summation to the indexes i, j running from 1 to n .

The conventional viewpoint is that the wave equation in an arbitrary Riemannian space V_n has the form (1). In this regard, A.S. Eddington wrote ([42], Chapter II, Section 30): “A transformation of coordinates does not alter the kind of space. Thus if we know by experiment that a potential

φ is propagated according to the law (3) in Galilean coordinates, it follows rigorously that it is propagated according to the law (6) in any system of coordinates in flat space-time; but it does not follow rigorously that it will be propagated according to (6) when an irreducible gravitational field is present which alters the kind of space-time. It is, however, a plausible suggestion that (6) may be the general law of propagation of φ in any kind of space-time; that is the suggestion which the principle of equivalence makes. Like all generalizations which are only tested experimentally in a particular case, it must be received with caution."

§ 2 Symmetry of the classical wave equation

Let us consider the problem of invariance of the wave equation with respect to the continuous group G_r of transformations in the form

$$\begin{aligned} x'^i &= x'^i(x, \varphi, a), \\ \varphi' &= \varphi'(x, \varphi, a), \end{aligned} \quad (7)$$

where $a = (a^1, \dots, a^r)$ is the parameter of the group G_r . One should not confuse the invariance of an equation with its covariance discussed in the previous section.

We will use the results on invariance properties of linear second-order partial differential equations obtained in [138]. Let us write an arbitrary linear homogeneous equation of the second order in the covariant form

$$F[\varphi] \equiv g^{ij}(x)\varphi_{,ij} + a^i(x)\varphi_{,i} + c(x)\varphi = 0, \quad (8)$$

where $\det \|g^{ij}\| \neq 0$; $i, j = 1, \dots, n$; $n > 2$. It is shown in [138] that the group G_r admitted by equation (8) has the generators of the form*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \sigma(x)\varphi\frac{\partial}{\partial \varphi}. \quad (9)$$

The coordinates ξ^i ($i = 1, \dots, n$) of the operator (9) are defined by the equations

$$\xi_{i,j} + \xi_{j,i} = \mu(x)g_{ij} \quad (i, j = 1, \dots, n), \quad (10)$$

$$(K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0 \quad (i, j = 1, \dots, n), \quad (11)$$

$$\xi^k H_{,k} + \mu H = 0, \quad (12)$$

*Only the essential part of the group is considered. Namely, we omit the transformations $\varphi' = k\varphi + \varphi_0(x)$, where $\varphi_0(x)$ is an arbitrary solution of equation (8) and $k = \text{const}$.

and the function $\sigma(x)$ is defined by

$$\sigma_{,i} = \frac{2-n}{4}\mu_{,i} - \frac{1}{2}(a_j\xi^j)_{,i} - \frac{1}{2}K_{ij}\xi^j \quad (i = 1, \dots, n). \quad (13)$$

Here the following notation has been introduced:

$$K_{ij} = a_{i,j} - a_{j,i} \quad (i, j = 1, \dots, n), \quad (14)$$

$$H = -2c + a^i_{,i} + \frac{1}{2}a^i a_i + \frac{n-2}{2(n-1)}R, \quad (15)$$

where R is the scalar curvature of the space V_n with the metric tensor g^{ij} , and the subscripts after the comma indicate the covariant differentiation in V_n . In what follows, we will use the transformations (see also [138])

$$x'^i = x'^i(x) \quad (i = 1, \dots, n) \quad (16)$$

$$F'[\varphi] = e^{-f(x)}F[\varphi e^{f(x)}], \quad (17)$$

$$F'[\varphi] = \Phi(x)F[\varphi]. \quad (18)$$

Equations (16)–(18) provide the general form of the transformations keeping the linearity and homogeneity of equation (8). These transformations are called *equivalence transformations* of equation (8). Equations obtained from each other by transformations (16), (17), (18) are called *equivalent equations*.

Recall that equations (10) define the group of conformal transformations in the space V_n . The group G_r admitted by (8) is a subgroup of the conformal group due to the additional equations (11) and (12). Since the conformal group in any space V_n of dimension $n > 2$ contains maximum

$$\frac{(n+1)(n+2)}{2} \quad (19)$$

parameters [46], we conclude that the dimension r of the group G_r is also bounded by the number (19).

The wave equation

$$\Delta\varphi \equiv \sum_{i=1}^n e_i \frac{\partial^2 \varphi_i}{(\partial x^i)^2} = 0 \quad (e_i = \pm 1) \quad (20)$$

in the flat space V_n is conformally invariant. Furthermore, the number of parameters of the conformal group in the flat space reaches its maximal value (19). Consequently, the symmetry group G_r for the wave equation (20) is the group of the maximal order (19). Moreover, we prove in the next section that any equation (8) admitting the group G_r of the maximal order $r = (n+1)(n+2)/2$ is equivalent to equation (20).

§ 3 Invariant equations in Riemannian spaces

Let us consider equation (8) as a basic second-order equation in V_n . Our aim is to single out the equations that are *conformally invariant*, i.e. admit the conformal group in V_n . It was mentioned in the previous section, that the group G_r admitted by equation (8) is a subgroup of the conformal group in V_n . Hence, our task is to find the equations for which G_r coincides with the conformal group. For example, if V_n is the flat space, such an equation is precisely (20). The latter equation has the form (6) in the covariant notation.

In general, in order to find the conformally invariant equation we have to determine the coefficients $a^i(x)$ ($i = 1, \dots, n$) and $c(x)$ of equation (8) so that the equations (11) and (12) are satisfied identically due to the equations (10). It is demonstrated in [138], that a necessary condition for the invariance of equation (8) with respect to the conformal group in V_n is provided by the equation

$$H = 0. \quad (21)$$

Therefore one has to investigate the equations (11) and (21) instead of (11) and (12). Note that the transformation (18) maps the space V_n to a conformal space. Consequently, we shall consider only the transformations (16) and (17) as equivalence transformations.

Theorem 7.1. Consider equation (8) in a conformally flat space V_n . If equation (8) is conformally invariant then it is equivalent to equation (2).

Proof. The metric tensor of any conformally flat space V_n can be reduced to the form

$$g^{ij} = \psi(x) \delta^{ij} \quad (i, j = 1, \dots, n) \quad (22)$$

by introducing certain number of imaginary coordinates (in accordance with the signature of g^{ij}) and using an appropriate change of coordinates (16). The order of the conformal group for the conformally flat space achieves its maximum value (19) and equations (10), due to (22), have the form

$$\xi_{i,j} + \xi_{j,i} = \frac{\mu(x)}{\psi(x)} \delta_{ij} \quad (i, j = 1, \dots, n). \quad (23)$$

The integrability conditions of equations (23) are satisfied identically [169]. Hence, the quantities ξ^i , $\xi_{i,j}$ and $\mu(x)$ are independent. Here $i = 1, \dots, n$ and $i < j$.

Let us show that, in this case, the condition of conformal invariance of equation (8) implies that the equations

$$K_{ij} = 0 \quad (i, j = 1, \dots, n) \quad (24)$$

hold. Indeed, let us rewrite equations (11) in the form

$$(K_{il}g^{lm}\delta_j^k - K_{jl}g^{lm}\delta_i^k)\xi_{m,k} + (K_{il,j} - K_{jl,i})\xi^l = 0. \quad (25)$$

Since the quantities ξ^l are independent, the equations (25) yield

$$(K_{il}g^{lm}\delta_j^k - K_{jl}g^{lm}\delta_i^k)\xi_{m,k} = 0 \quad (i, j = 1, \dots, n). \quad (26)$$

Furthermore, we have from equations (23) that

$$\xi_{1,1} = \xi_{2,2} = \dots = \xi_{n,n} = \frac{1}{2} \frac{\mu}{\psi}. \quad (27)$$

The remaining quantities $\xi_{m,k}$ ($m \neq k$) are not connected with the function $\mu(x)$. Keeping this in mind and invoking (22) we single out in equations (26) all terms containing μ . Using the equations $K_{ji} = -K_{ij}$, one obtains consecutively

$$\begin{aligned} & \sum_{m=1}^n (K_{il}g^{lm}\delta_j^m - K_{jl}g^{lm}\delta_i^m)\xi_{m,m} \\ &= \frac{1}{2} \frac{\mu}{\psi} \sum_{m=1}^n (K_{il}\psi\delta^{lm}\delta_j^m - K_{jl}\psi\delta^{lm}\delta_i^m) \\ &= \frac{n}{2} \mu (K_{ij} - K_{ji}) = n\mu K_{ij}. \end{aligned} \quad (28)$$

Since μ is arbitrary, equations (28) lead to (24).

It follows from equations (24) that

$$a_i = b_{,i} \quad (i = 1, \dots, n) \quad (29)$$

with a certain function $b(x)$. Now the equivalence transformation (17) with $f(x) = \frac{1}{2} b(x)$ maps our equation to an equivalent equation with

$$a_i = 0 \quad (i = 1, \dots, n). \quad (30)$$

Therefore, we can assume that equation (8) has the following form:

$$g^{ij}(x)\varphi_{,ij} + c(x)\varphi = 0. \quad (31)$$

This equation is written in the covariant form, and hence it holds in an arbitrary coordinate system. Therefore, it follows from equation (21) and the definition of the function H that

$$c(x) = \frac{n-2}{4(n-1)} R. \quad (32)$$

Substituting (32) into (31) one obtains equation (2). The theorem is proved.

Theorem 7.2. For any Riemannian space V_n , equation (2) is invariant with respect to the group of conformal transformations in the space V_n .

Proof. Since equations (21) and (24) are satisfied for equation (2), the equations (11) and (12) hold identically and provide the conformal invariance of equation (2). The theorem is proved.

Theorems 7.1 and 7.2 guarantee that equation (2) is the only (up to the equivalence transformation (17)) conformally invariant equation in any Riemannian space V_n . Thus, the following definition is suggested.

Definition 7.1. The equation

$$\Delta\varphi \equiv g^{ij}\varphi_{,ij} + \frac{n-2}{4(n-1)}R\varphi = 0 \quad (33)$$

is called the *invariant wave equation*.

Remark 7.1. In this definition, the expression *invariant* emphasizes not only the conformal invariance of equation (33), but rather the fact that (33) is a *universal conformally invariant equation* for any space V_n . The usual wave equation (3), unlike (33), does not possess this property.

Equation (33) has one more remarkable property. Namely, the invariant wave equations in conformal spaces are connected with each other by means of the equivalence transformation (18). Specifically, the following theorem holds.

Theorem 7.3. Let two spaces V_n and \bar{V}_n be conformal to each other and have metric tensors

$$g_{ij} \quad \text{and} \quad \bar{g}_{ij} = e^{2\theta(x)}g_{ij} \quad (i, j = 1, \dots, n), \quad (34)$$

respectively. Then

$$\bar{\Delta}\varphi = e^{-\frac{n+2}{2}\theta}\Delta(\varphi e^{\frac{n-2}{2}\theta}), \quad (35)$$

where $\Delta\varphi$ and $\bar{\Delta}\varphi$ are given by the formula (33) for V_n and \bar{V}_n , respectively.

Proof. Let us introduce the notation $\varphi_i = \frac{\partial\varphi}{\partial x^i}$, $\varphi_{ij} = \frac{\partial^2\varphi}{\partial x^i\partial x^j}$, and likewise for the function $\theta(x)$. Equation (33) is written in the form

$$\Delta\varphi = g^{ij}(\varphi_{ij} - \Gamma_{ij}^k\varphi_k) + \frac{n-2}{4(n-1)}R\varphi, \quad (36)$$

where Γ_{ij}^k are the Christoffel symbols.

The scalar curvatures and the Christoffel symbols of V_n and \bar{V}_n are connected by the relations

$$R = e^{2\theta} [\bar{R} - 2(n-1)g^{ij}(\theta_{ij} - \bar{\Gamma}_{ij}^k \theta_k) + (n-1)(n-2)\bar{g}^{ij}\theta_i\theta_j], \quad (37)$$

and

$$\Gamma_{ij}^k = \bar{\Gamma}_{ij}^k - \delta_i^k \theta_j - \delta_j^k \theta_i + \bar{g}_{ij} \bar{g}^{kl} \theta_l, \quad (38)$$

respectively (see, e.g. [46]). Using the equations (37) and (38) one obtains:

$$\begin{aligned} e^{-\frac{n+2}{2}\theta} \Delta(\varphi e^{\frac{n-2}{2}\theta}) &= e^{-2\theta} g^{ij} \left[\varphi_{ij} + \frac{n-2}{2} \varphi_i \theta_j + \frac{n-2}{2} \varphi_j \theta_i + \frac{n-2}{2} \varphi \theta_{ij} \right. \\ &\quad \left. + \left(\frac{n-2}{2} \right)^2 \varphi \theta_i \theta_j - \Gamma_{ij}^k \varphi_k - \frac{n-2}{2} \varphi \Gamma_{ij}^k \theta_k \right] + \frac{n-2}{4(n-1)} e^{-2\theta} R \varphi \\ &= \bar{g}^{ij} \left[\varphi_{ij} - \bar{\Gamma}_{ij}^k \varphi_k + \frac{n}{2} (\varphi_i \theta_j + \varphi_j \theta_i) - \bar{g}_{ij} \bar{g}^{kl} \theta_l \varphi_k \right] \\ &\quad + \frac{n-2}{2} \varphi \bar{g}^{ij} (\theta_{ij} + \frac{n-2}{2} \theta_i \theta_j - \Gamma_{ij}^k \theta_k + 2\theta_i \theta_j - \bar{g}_{ij} \bar{g}^{kl} \theta_l \theta_k) \\ &\quad + \frac{n-2}{4(n-1)} [\bar{R} - 2(n-1)\bar{g}^{ij}(\theta_{ij} - \bar{\Gamma}_{ij}^k \theta_k) + (n-1)(n-2)\bar{g}^{ij}\theta_i\theta_j] \varphi \\ &= \bar{g}^{ij} (\varphi_{ij} - \bar{\Gamma}_{ij}^k \varphi_k) + \frac{n-2}{4(n-1)} \bar{R} \varphi = \bar{\Delta} \varphi, \end{aligned}$$

thus completing the proof.

This theorem allows one to find solutions to wave equations (33) in the whole class of conformal to each other spaces V_n , provided that a solution to the wave equation in at least one of the spaces is known.

Remark 7.2. Although transformation (18) relates any spaces conformal to each other, it does not mean that it relates any equations of the form (8) in conformal spaces. For instance, equations (1) in conformal spaces cannot be related by any transformation (18). Therefore, Theorem 7.3 indicates once more that equation (33) is special among all equations of the type (8).

Theorem 7.4. Any linear homogeneous second-order differential equation (8) admitting a group G_r of the maximal order $r = (n+1)(n+2)/2$ is equivalent to equation (20) of the appropriate signature.

Proof. It follows from the condition of the theorem that equation (8) is a conformally invariant and that the conformal group of the corresponding Riemannian space V_n has the maximal order (19). It is known [169] that any space V_n possessing the conformal group of the maximal order (19) is conformally flat. Therefore the statement of the theorem follows from Theorems 7.1 and 7.3.

§ 4 Application in physics

In the physically significant case $n = 4$, I will use the common symbol \square for the wave operator and write equation (33) in a Riemannian space V_4 in the form

$$\square\varphi \equiv g^{ij}\varphi_{,ij} + \frac{1}{6}R\varphi = 0. \quad (39)$$

It follows from the condition (35) that if the equation

$$\square\varphi = 0 \quad (40)$$

is valid in the space V_4 then the equation

$$\overline{\square}\bar{\varphi} = 0 \quad (41)$$

holds in the conformal space \overline{V}_4 as well, where

$$\bar{\varphi} = \varphi e^{-\theta}. \quad (42)$$

As it was mentioned in [71], these formulae are useful in the theory of the red shift.

Equation (39) and formulae (40)–(42) are considered in [147], they furnish interesting physical results.

The present paper was discussed with Professor L.V.Ovsyannikov. Professor A.D. Aleksandrov made several valuable observations, and Professor A. Trautman recently draw my attention to the paper [147]. I express them my deepest gratitude.

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Paper 8

Invariant variational problems and conservation laws (remarks on Noether's theorem)

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Noether's theorem on the conservation laws for invariant variational problems is generalized. All the conservation laws for the Dirac equations are found in an illustration of the generalization.

Noether's theorem [133] gives a convenient procedure for obtaining conservation laws for equations of mathematical physics arising from a variational principle. According to this theorem, the existence of r independent conservation laws follows from the invariance of the variational integral with respect to a continuous r -parameter transformation group. Examples show, however, that the converse is not true: existence of conservation laws does not imply an invariance of a variational integral with respect to the corresponding group. Therefore, the procedure for obtaining the conservation laws, based on Noether's theorem, is sometimes not sufficiently general (even with account of divergence transformations [20], [65]).

We propose in this paper a certain modification of the concept of the invariance of the variational integral; this leads to conservation laws of the type (12). Conversion of Noether's theorem is proved for non-degenerate variational problems. The general results obtained are illustrated for the Dirac equations. Interestingly, the well-known Pauli group [146] for the

Dirac equations does not satisfy the conditions of Noether's theorem.

§ 1 Noether's theorem

For simplicity, we restrict the discussion to variational integrals of the form

$$\int_{\Omega} \mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) d\omega, \quad (1)$$

where $x = (x^1, \dots, x^n)$ are independent variables, $\psi = (\psi^1, \dots, \psi^m)$, are dependent variables, and $\frac{\partial \psi}{\partial x}$ denotes the set of first derivatives $\psi_i^k = \frac{\partial \psi^k}{\partial x^i}$ ($i = 1, \dots, n; k = 1, \dots, m$). This is not, however, a fundamental restriction. All the discussion can be easily extended to arbitrary variational integrals in which the Lagrangian \mathcal{L} depends on derivatives of any order.

We denote by G_r a continuous r -parameter transformation group:

$$x'^i = f^i(x, \psi, a) \quad (i = 1, \dots, n), \quad (2)$$

$$\psi'^k = F^k(x, \psi, a) \quad (k = 1, \dots, m), \quad (3)$$

where $a = (a^1, \dots, a^r)$ is the group parameter. It is assumed that for some value a_0 of the group parameter the transformations (2) and (3) are identical, i.e.,

$$f^i(x, \psi, a_0) = x^i \quad (i = 1, \dots, n), \quad F^k(x, \psi, a_0) = \psi^k \quad (k = 1, \dots, m).$$

As usual, we will write the infinitesimal operators of the group G_r in the form [45], [138]

$$X_{\alpha} = \xi_{\alpha}^i \frac{\partial}{\partial x^i} + \eta_{\alpha}^k \frac{\partial}{\partial \psi^k} \quad (\alpha = 1, \dots, r), \quad (4)$$

where

$$\xi_{\alpha}^i(x, \psi) = \left. \frac{\partial f^i}{\partial a^{\alpha}} \right|_{a=a_0}, \quad \eta_{\alpha}^k(x, \psi) = \left. \frac{\partial F^k}{\partial a^{\alpha}} \right|_{a=a_0}. \quad (5)$$

Here and below, the Einstein convention of the summation of repeated subscripts and superscripts is adopted.

Consider given functions

$$\psi^k = \psi^k(x) \quad (k = 1, \dots, m). \quad (6)$$

Substituting (6) into (2), solving the resulting equations for the variables x^i ($i = 1, \dots, n$), and using (3), we obtain the functions

$$\psi'^k = \psi'^k(x') \quad (k = 1, \dots, m)$$

(it should be kept in mind that these functions involve also the group parameter). If the following equation holds for all functions (6) and all transformations of the group G_r :

$$\int_{\Omega} \mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) d\omega = \int_{\Omega'} \mathcal{L} \left(x', \psi', \frac{\partial \psi'}{\partial x'} \right) d\omega',$$

where the integration region Ω' is obtained from Ω by transformations (2) on account* of (6), the variational integral (1) is called invariant with respect to the group G_r . Reducing the integration over the region Ω' to an integration over Ω , and taking into account the arbitrariness of the region Ω , one can rewrite the condition for the invariance of integral (1) in the form

$$\mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) d\omega = \mathcal{L} \left(x', \psi', \frac{\partial \psi'}{\partial x'} \right) J \left(\frac{x'}{x} \right) d\omega, \quad (7)$$

where $J(x'/x) = \det ||D_i f^j||$ is the Jacobian of the coordinate transformation. Equation (7) shows that the function $\mathcal{L}(x, \psi, \frac{\partial \psi}{\partial x})d\omega$ is an invariant of the group \tilde{G}_r obtained from the group G_r by a prolongation of the transformations (2) and (3) to the derivatives ψ_i^k and the volume element $d\omega$. We can express the necessary and sufficient condition for this invariance in terms of infinitesimal operators (4). We will calculate for this purpose the infinitesimal operators of the group \tilde{G}_r . We seek them in the form

$$\tilde{X}_\alpha = X_\alpha + \zeta_{\alpha i}^k \frac{\partial}{\partial \psi_i^k} + \mu_\alpha \frac{\partial}{\partial d\omega} \quad (\alpha = 1, \dots, r).$$

The quantities $\zeta_{\alpha i}^k$ ($i = 1, \dots, n$; $k = 1, \dots, m$; $\alpha = 1, \dots, r$) are given by the well-known prolongation formulas [138]

$$\zeta_{\alpha i}^k = D_i(\eta_\alpha^k) - \psi_j^k D_i(\xi_\alpha^j),$$

where D_i is the operator for total differentiation with respect to x^i :

$$D_i = \frac{\partial}{\partial x^i} + \psi_i^k \frac{\partial}{\partial \psi^k} + \psi_{ij}^k \frac{\partial}{\partial \psi_j^k} + \dots \quad (i = 1, \dots, n).$$

The coefficients μ_α ($\alpha = 1, \dots, r$) are determined by the general rule (5) from the transformation equation for the volume element,

$$d\omega' = J(x'/x)d\omega.$$

*The transformation (2) of the quantities x^i ($i = 1, \dots, n$) also depends on the functions ψ^k ($k = 1, \dots, m$). Therefore, it is meaningful to discuss transformation of the coordinates x^i ($i = 1, \dots, n$) only on account of (6).

The rule for determinant differentiation, with account of (2) and (5), yields:

$$\left. \frac{\partial J(x'/x)}{\partial a^\alpha} \right|_{a=a_0} - \frac{\partial}{\partial a^\alpha} (\det ||D_i f^j(x, \psi, a)||) \Big|_{a=a_0} = D_i(\xi_\alpha^i),$$

and therefore

$$\mu_\alpha = \left. \frac{\partial d\omega'}{\partial a^\alpha} \right|_{a=a_0} = \left. \frac{\partial J(x'/x)}{\partial a^\alpha} \right|_{a=a_0} d\omega = D_i(\xi_\alpha^i) d\omega.$$

The test for the invariance [138] of the function $\mathcal{L}(x, \psi, \frac{\partial \psi}{\partial x}) d\omega$ is written

$$\tilde{X}_\alpha(\mathcal{L} d\omega) \equiv \left(\xi_\alpha^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta_\alpha^k \frac{\partial \mathcal{L}}{\partial \psi^k} + \zeta_{\alpha i}^k \frac{\partial \mathcal{L}}{\partial \psi_i^k} \right) d\omega + \mathcal{L} \mu_\alpha = 0.$$

Invoking the prolongation formulae found above and arbitrariness of $d\omega$ this equation can be written as

$$\xi_\alpha^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta_\alpha^k \frac{\partial \mathcal{L}}{\partial \psi^k} + [D_i(\eta_\alpha^k) - \psi_j^k D_i(\xi_\alpha^j)] \frac{\partial \mathcal{L}}{\partial \psi_i^k} + \mathcal{L} D_i(\xi_\alpha^i) = 0, \quad (8)$$

where $\alpha = 1, \dots, r$. After certain transformations of the left side of (8), one arrives the following necessary and sufficient condition for the invariance of the integral (1) with respect to the group G_r :

$$(\eta_\alpha^k - \psi_j^k \xi_\alpha^j) \frac{\partial \mathcal{L}}{\partial \psi_i^k} + D_i(A_\alpha^i) = 0 \quad (\alpha = 1, \dots, r), \quad (9)$$

where

$$\frac{\delta \mathcal{L}}{\delta \psi^k} = \frac{\partial \mathcal{L}}{\partial \psi^k} - D_i \left(\frac{\partial \mathcal{L}}{\partial \psi_i^k} \right)$$

and

$$A_\alpha^i = (\eta_\alpha^k - \psi_j^k \xi_\alpha^j) \frac{\partial \mathcal{L}}{\partial \psi_i^k} + \mathcal{L} \xi_\alpha^i \quad (i = 1, \dots, n; \alpha = 1, \dots, r). \quad (10)$$

This was established by E. Noether [133]. An important result, known as Noether's theorem, follows.

Theorem 8.1. (Noether's theorem) Let the variational integral (1) be invariant with respect to a group G_r with operators (4). Then the divergences of r linearly independent vectors A_α ($\alpha = 1, \dots, r$) with components (10) vanish for any solution of the Euler equations,

$$\frac{\delta \mathcal{L}}{\delta \psi^k} = 0, \quad (11)$$

i.e., the following equations hold:

$$D_i(A_\alpha^i) \Big|_{(11)} = 0, \quad \alpha = 1, \dots, r. \quad (12)$$

The existence of r invariants (i.e., quantities which do not depend on the time $t = x^n$) of Eqs. (11) follows from Equation (12), the invariants being defined by [27] $I_\alpha = \int A_\alpha^n dx^1 \dots dx^{n-1}$. Equations (12) are usually termed conservation laws. Various conservation laws obtained via Noether's theorem can be found in [133], [20], [65], [27], [47].

§ 2 Remarks on Noether's theorem

The necessary condition for the invariance of integral (1) with respect to the group G_r is the existence of the following properties:

A. The group G_r transforms any extremum of integral (1) into some extremum of the same integral. This means that the Euler equations (11) are invariant with respect to the group G_r .

B. The conservation laws (12) hold for all operators (4) of the group G_r and for all solutions of the Euler equations (11).

Properties A and B do not in general constitute a sufficient condition for the invariance of the integral (1) with respect to the group G_r . This means that not all conservation laws (12) can be found from Noether's theorem. Therefore, we seek a principle regarding the invariant properties of the variational integral (1), and based on the same grounds as Noether's theorem, which will permit us to find all the conservation laws (12). The following theorem establishes this principle.

Theorem 8.2. Let the group G_r satisfy condition A. In order that condition B holds, it is necessary and sufficient that the value of the variational integral (1) be invariant with respect to the group G_r at the extrema.

Proof. Invariance of the value of the variational integral at the extrema with respect to the group G_r means that Equation (7) holds for all functions (6) which are solutions of Equation (11). It is more convenient here to rewrite Equation (7) without the factor $d\omega$:

$$\mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) = \mathcal{L} \left(x', \psi', \frac{\partial \psi'}{\partial x'} \right) J \left(\frac{x'}{x} \right). \quad (13)$$

Let us consider the group \tilde{G}_r of point transformations of the space of the variables $x^i, \psi^k, \psi_i^k, \psi_{ij}^k$ ($i, j = 1, \dots, n; k = 1, \dots, m$) and L , given by Equations (2) and (3), along with their prolongation [138] to the first and second derivatives of ψ_i^k and ψ_{ij}^k , and by the equation

$$L' = J^{-1} \left(\frac{x'}{x} \right) L. \quad (14)$$

Validity of Equation (13) for all the extrema and the condition (A) imply that the manifold given by the equations (11), $\frac{\delta \mathcal{L}}{\delta \psi^h} = 0$, and

$$L = \mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) \quad (15)$$

is an invariant manifold for \tilde{G}_r . The quantity L does not appear in Equation (11), so Equation (15) does not affect the invariance condition of these equations with respect to the group \tilde{G}_r . We can thus assume immediately, as was done in the conditions of the theorem, that a group with respect to which Equations (11) are invariant is chosen as the group G_r . Derivatives of order no higher than the first appear in Equation (15), so that the group G_r should be prolonged only to the first derivatives ψ_i^k ($i = 1, \dots, n$; $k = 1, \dots, m$). The operators of the group \tilde{G}_r will be written in the form

$$\tilde{X}_\alpha = X_\alpha + \zeta_{\alpha i}^k \frac{\partial}{\partial \psi_i^k} + M_\alpha \frac{\partial}{\partial L}.$$

The coefficients $\zeta_{\alpha i}^k$ can be found from the prolongation formulas [138], and M_α can be found by letting the operator $\frac{\partial}{\partial a^\alpha}|_{a=a_0}$ act on Equation (14), $M_\alpha = \frac{\partial L'}{\partial a^\alpha}|_{a=a_0} = -LD_i(\xi_\alpha^i)$. We thus have

$$\tilde{X}_\alpha = \xi_\alpha^i \frac{\partial}{\partial x^i} + \eta_\alpha^k \frac{\partial}{\partial \psi^k} + [D_i(\eta_\alpha^k) - \psi_j^k D_i(\xi_\alpha^j)] \frac{\partial}{\partial \psi_i^k} - LD_i(\xi_\alpha^i) \frac{\partial}{\partial L}. \quad (16)$$

Since Equation (11) are invariant with respect to the group G_r , a necessary and sufficient condition for the invariance of the manifold given by Equations (11) and (15) is that the following equalities be true:

$$\tilde{X}_\alpha \left(\mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) - L \right) \Big|_{(11), (15)} = 0 \quad (\alpha = 1, \dots, r). \quad (17)$$

The left sides of Equations (17) imply that the expressions in braces are calculated on the manifold given by Equations (11) and (15). Because of Equation (16), we have

$$\begin{aligned} \tilde{X}_\alpha (\mathcal{L} - L) \Big|_{(15)} &= \mathcal{L} D_i(\xi_\alpha^i) + \xi_\alpha^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta_\alpha^k \frac{\partial \mathcal{L}}{\partial \psi^k} \\ &+ [D_i(\eta_\alpha^k) - \psi_j^k D_i(\xi_\alpha^j)] \frac{\partial \mathcal{L}}{\partial \psi_i^k}, \quad (\alpha = 1, \dots, r). \end{aligned} \quad (18)$$

Simplifying Equations (18) (as we did with Equations (8)) we obtain

$$\tilde{X}_\alpha (\mathcal{L} - L) \Big|_{(15)} = (\eta_\alpha^k - \psi_j^k \xi_\alpha^j) \frac{\delta \mathcal{L}}{\delta \psi^k} + D_i(A_\alpha^i), \quad (\alpha = 1, \dots, r). \quad (19)$$

Equation (17) thus has the form (12). The theorem is proved.

In Theorem 8.2, in contrast with Noether's theorem, we are not interested in the behavior of the variational integral (1) for the functions (6) which are not extrema of the integral (1). It may accordingly turn out that not all the conservation laws (12) obtained from Theorem 8.2 are independent. This means that the vectors A_α ($\alpha = 1, \dots, r$) with the components (10) may turn out to be linearly dependent. However, the number of linearly independent vectors will in any case be no less than in Noether's theorem, since all the conservation laws (12) obtained through Noether's theorem satisfy the conditions of Theorem 8.2. In this sense, Theorem 8.2 is more general than Noether's theorem.

Using Theorem 8.2, one can show that the converse of Noether's theorem holds for non-degenerate variational problems. We call a variational problem *non-degenerate* when all the Euler equations in the system (11) are of second order. Otherwise (i.e., when some of the equations of system (11) can be written as first-order equations), the variational problem will be called *degenerate*.

Theorem 8.3. Assume properties A and B hold for a non-degenerate variational problem. Then the variational integral (1) is invariant with respect to the group G_r .

Proof. Invariance of the variational integral (1) with respect to the group G_r is equivalent to the invariance of the manifold given by Equation (15) with respect to the group \tilde{G}_r with the operators (16). Therefore, in order to prove the theorem, we must show that the equations

$$\tilde{X}_\alpha(\mathcal{L} - L)\Big|_{(15)} = 0, \quad (\alpha = 1, \dots, r) \quad (20)$$

hold identically for all variables x^i, ψ^k, ψ_i^k ($i = 1, \dots, n; k = 1, \dots, m$) for all non-degenerate variational problems as a result of conditions A and B.

Because of A and B, we find from (19) that

$$\tilde{X}_\alpha(\mathcal{L} - L)\Big|_{(15), (11)} = (\eta_\alpha^k - \psi_j^k \xi_\alpha^j) \frac{\delta \mathcal{L}}{\delta \psi^k}\Big|_{(11)} + D_i(A_\alpha^i)\Big|_{(11)} = 0. \quad (21)$$

On the other hand, the right sides of Equations (18) do not depend on the second derivative of ψ_{ij}^k ($i, j = 1, \dots, n; k = 1, \dots, m$). Therefore, the left sides of these equalities, i.e., the expressions $\tilde{X}_\alpha(\mathcal{L} - L)\Big|_{(15)}$, are also independent of the second derivatives. In addition, because of the non-degenerate nature of the variational problem, the Euler equations (11) contain no restrictions of any kind on the quantities x^i, ψ^k, ψ_i^k . Therefore, we have

$$\tilde{X}_\alpha(\mathcal{L} - L)\Big|_{(11), (15)} \equiv \tilde{X}_\alpha(\mathcal{L} - L)\Big|_{(15)}, \quad (22)$$

and hence Equations (20) follow from (21). The theorem is proved.

Remark 8.1. The proofs of both these theorems are valid for variational problems in which the Lagrangian \mathcal{L} depends on derivatives of order up to N . Instead of the first derivatives here, we must take derivatives of the N -th order; instead of second derivatives, we must take derivatives of the $(N + 1)$ -th order; and we must prolong the group G_r the corresponding number of times.

§ 3 The Dirac equations

The Dirac equations

$$\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi = 0, \quad \frac{\partial \bar{\psi}}{\partial x^k} \gamma^k - m\bar{\psi} = 0 \quad (23)$$

can be written as the Euler equations (11) with the Lagrangian

$$L = \frac{1}{2} \left[\bar{\psi} \left(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi \right) - \left(\frac{\partial \bar{\psi}}{\partial x^k} \gamma^k - m\bar{\psi} \right) \psi \right]. \quad (24)$$

Here the usual notation is adopted: ψ is the column vector with components ψ^k ($k = 1, 2, 3, 4$) and $\bar{\psi}$ is the row vector (with components $\bar{\psi}_k$) defined by

$$\bar{\psi} = \psi^T \gamma^4. \quad (25)$$

The asterisk denotes complex conjugation, and T denotes transposition. We will use the following representation of the matrices γ^k :

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^3 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \gamma^4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Since Equations (23) are linear, they admit the transformations

$$\psi' = \psi + a\varphi(x), \quad \bar{\psi}' = \bar{\psi} + a\bar{\varphi}(x), \quad (26)$$

where the column vector $\varphi(x)$ solves the Dirac equations (23), $\bar{\varphi}(x)$ is calculated from $\varphi(x)$ according to Equation (25), and a is an arbitrary real

number (the group parameter). The one-parameter transformation group (26) has the infinitesimal operator

$$X_\varphi = \varphi^k(x) \frac{\partial}{\partial \psi^k} + \bar{\varphi}_k(x) \frac{\partial}{\partial \bar{\psi}_k}. \quad (27)$$

The Dirac equations have an infinite number of linearly independent solutions; therefore the algebra of Lie operators (27) is infinite-dimensional for all possible solutions $\varphi(x)$ of the Dirac equations, and the transformations (26) form an infinite-parameter group G_∞ . In addition to the transformations (26), the Dirac equations (23) are invariant with respect to the following transformations.

Equations (23) with $m = 0$ are invariant with respect to the continuous group with the infinitesimal operators

$$\bar{X} = X + (S\psi)^k \frac{\partial}{\partial \psi^k} + (\bar{\psi}\bar{S})_k \frac{\partial}{\partial \bar{\psi}_k}, \quad (28)$$

where $X = \xi^k(x) \frac{\partial}{\partial x^k}$ runs over the following system of generators of the 15-parameter conformal group:

$$X_k = \frac{\partial}{\partial x^k} \quad (k = 1, 2, 3, 4), \quad (29)$$

$$X_{kl} = x^l \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^l} \quad (k < l; \quad k, l = 1, 2, 3, 4), \quad (30)$$

$$X_0 = x^k \frac{\partial}{\partial x^k}, \quad (31)$$

$$Y_k = (2x^k x^l - |x|^2 \delta^{kl}) \frac{\partial}{\partial x^l} \quad (k, l = 1, 2, 3, 4). \quad (32)$$

Here $|x|^2 = \sum_{k=1}^4 (x^k)^2$, the matrix S is expressed via the coordinates ξ^k of the operators (29)-(32) by means of the equation [70]

$$S = \frac{1}{S} \sum_{k,l=1}^4 \frac{\partial \xi^k}{\partial x^l} (\gamma^k \gamma^l - \gamma^l \gamma^k - 3\delta^{kl}), \quad (33)$$

and

$$\bar{S} = \gamma^4 S^T \gamma^4. \quad (34)$$

Equations (23) are also invariant with respect to the eight-parameter group composed by the following one-parameter transformation groups with real

parameters a [74] ($\bar{\psi}'$ is obtained from ψ' by means of Equation (25)):

$$\psi' = \psi e^a, \quad (35)$$

$$\psi' = \psi e^{-ia}, \quad (36)$$

$$\psi' = \psi \cosh a + \gamma^4 \gamma^2 \bar{\psi}^T \sinh a, \quad (37)$$

$$\psi' = \psi \cosh a + i \gamma^4 \gamma^2 \bar{\psi}^T \sinh a, \quad (38)$$

$$\psi' = \psi e^{ia\gamma^5}, \quad \gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4, \quad (39)$$

$$\psi' = \psi e^{-a\gamma^5} \quad (40)$$

$$\psi' = \psi \cos a + \gamma^3 \gamma^1 \bar{\psi}^T \sin a, \quad (41)$$

$$\psi' = \psi \cos a + i \gamma^3 \gamma^1 \bar{\psi}^T \sin a. \quad (42)$$

Transformations* (28)-(42), along with (26), form the largest group $G_{23+\infty}$ admitted by Equations (23) in the sense of S. Lie.

In the case of a non-vanishing mass ($m \neq 0$), the largest group admitted in the sense of S. Lie by Equations (23) is the group $G_{14+\infty}$, formed by transformations (26), (28)-(30), and (35)-(38).

We turn to the question of the conservation laws for the Dirac equations, for the case of zero mass. For the case $m \neq 0$ the discussion is similar, and is omitted.

The Lagrangian (24) vanishes on solutions of Dirac's equations (23) since the value of the variational integral at the extrema is zero. It can be verified that all transformations of the group $G_{23+\infty}$ retain the zero value of the variational integral for (24) at the extrema. Therefore, it follows from Theorem 8.2 that all transformations of this group provide certain conservation laws (12). Before we calculate these conservation laws, we consider the question of which of them can be obtained from Noether's theorem. First, we note that the variational problem is degenerate in this case, so Theorem 8.3 does not apply. A check shows that transformations (26), (35), (40), (41), and (42) do not satisfy the condition of Noether's theorem, i.e., they do not leave invariant the variational integral for the Lagrangian (24) for all functions ψ . For example, with transformation (40), we find

$$\begin{aligned} \mathcal{L} \left(x', \psi', \frac{\partial \psi'}{\partial x'} \right) &= (\cosh^2 a + \sinh^2 a) \mathcal{L} \left(x, \psi, \frac{\partial \psi}{\partial x} \right) \\ &+ \cosh a \sinh a \left(\bar{\psi} \gamma^5 \gamma^k \frac{\partial \psi}{\partial x^k} + \frac{\partial \bar{\psi}}{\partial x^k} \gamma^k \gamma^5 \psi \right), \end{aligned}$$

*For brevity, the operators (28) are called transformations. We mean, of course, the transformations obtained from these operators by solving the Lie equations [45], [138].

from which we see that invariance holds only for functions ψ solving the Dirac equations (23). The remaining 19-parameter subgroup of the group $G_{23+\infty}$ satisfies Noether's theorem*, so that the latter yields only 19 conservation laws for the Dirac equations (23) with $m = 0$.

Equation (10) yields the following vectors for which the conservation laws (12) hold. In order to indicate the transformations of $G_{23+\infty}$ associated with conserved vectors, the corresponding labels with prime are used. The vectors presented below are linearly independent and give an infinite number of conservation laws for the Dirac equations (23) with $m = 0$:

$$A_\varphi^k = \bar{\psi}\gamma^k\varphi - \bar{\varphi}\gamma^k\psi \quad (27')$$

$$A_l^k = \frac{1}{2} \left[\frac{\partial \bar{\psi}}{\partial x^l} \gamma^k \psi - \bar{\psi} \gamma^k \frac{\partial \psi}{\partial x^l} + \delta_l^k \left(\bar{\psi} \gamma^j \frac{\partial \psi}{\partial x^j} - \frac{\partial \bar{\psi}}{\partial x^j} \gamma^j \psi \right) \right], \quad (29')$$

$$A_{lm}^k = \frac{1}{4} [\bar{\psi}(\gamma^k \gamma^l \gamma^m + \gamma^l \gamma^m \gamma^k) \psi] + x^m A_l^k - x^l A_m^k, \quad (30')$$

$$A_0^k = x^l A_l^k, \quad (31')$$

$$B_l^k = 2x^m A_{ml}^k + |x|^2 A_l^k, \quad (32')$$

$$C_i^k = -i\bar{\psi}\gamma^k\psi, \quad (36')$$

$$C_2^k = \frac{1}{2} (\bar{\psi}\gamma^k\gamma^4\gamma^2\bar{\psi}^T - \psi^T\gamma^4\gamma^2\gamma^k\psi), \quad (37')$$

$$C_3^k = \frac{i}{2} (\bar{\psi}\gamma^k\gamma^4\gamma^2\bar{\psi}^T + \psi^T\gamma^4\gamma^2\gamma^k\psi), \quad (38')$$

$$C_4^k = i\bar{\psi}\gamma^k\gamma^5\psi, \quad (39')$$

$$C_5^k = \frac{1}{2} (\bar{\psi}\gamma^k\gamma^3\gamma^1\bar{\psi}^T + \psi^T\gamma^3\gamma^1\gamma^k\psi), \quad (41')$$

$$C_6^k = \frac{i}{2} (\bar{\psi}\gamma^k\gamma^3\gamma^1\bar{\psi}^T - \psi^T\gamma^3\gamma^1\gamma^k\psi). \quad (42')$$

For the transformations of (35) and (40), the corresponding vectors are equal to zero. This agrees with the remark pertaining to Theorem 8.2 about the possible linear dependence among the vectors obtained.

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*It is not necessary to seek finite transformations of the group $G_{23+\infty}$ in order to check these assertions. It is sufficient to check the equations (9) for infinitesimal operators.

Paper 9

Sur le problème de J. Hadamard relatif à la diffusion des ondes

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1. Le problème de la diffusion des ondes [61] est une partie essentielle de la théorie des équations hyperboliques du second ordre

$$\sum_{i,j=1}^n g^{ij}(x)u_{ij} + \sum_{i=1}^n b^i(x)u_i + c(x)u = 0. \quad (1)$$

On dit que l'équation (1) n'entraîne pas la diffusion des ondes si la solution du problème de Cauchy au point $x = (x^1, \dots, x^n)$ dépend seulement des valeurs initiales dans un voisinage suffisamment petit de l'intersection du support des valeurs initiales et du cône caractéristique ayant x pour sommet. Dans le cas de n impair la diffusion des ondes a toujours lieu [61]. L'équation classique de propagation des ondes pour n pair et $n \geq 4$ présente un exemple remarquable d'une équation qui n'entraîne jamais la diffusion des ondes.

C'est J. Hadamard qui a posé le problème de trouver toutes les équations de forme (1) qui n'entraînent jamais la diffusion des ondes ([63], [64], [40], [11], [35]). Un résultat remarquable a été obtenu dans cette direction par Myron Mathisson [128]. Il a prouvé que dans le cas où $n = 4$ et $g^{ij} =$

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Cte $(i, j = 1, \dots, 4)$, toute équation de forme (1) qui n'entraîne jamais de diffusion des ondes peut être transformée en l'équation classique de la propagation des ondes

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0 \quad (2)$$

au moyen d'un changement des variables et de la fonction inconnue. D'autre part, Karl Stellmacher [164] a montré que dans le cas $n \geq 6$ il existe des équations de forme (1) qui n'entraînent jamais de diffusion des ondes et qui ne se réduisent pas à l'équation classique de la propagation des ondes.

Dans le cas $n = 4$ on ne connaissait jusqu'à présent aucune équation ne se réduisant pas à l'équation (2) et n'entraînant pas de diffusion des ondes. Mais il existe de telles équations. En voici une:

2. Soit $0 < f(\sigma) \in C^1(R)$. Considérons le problème de Cauchy

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - u_{zz} = 0; \quad (3)$$

$$u|_{t=0} = 0, \quad (4)$$

$$u_t|_{t=0} = \varphi(x, y, z). \quad (5)$$

La restriction (4) ne joue pas de rôle essentiel. En effet, un problème de Cauchy pour l'équation (3) à données initiales arbitraires peut être réduit au problème (3)-(5). Donc, l'existence ou bien l'absence de la diffusion des ondes ne dépend pas de la condition (4). Pour l'équation (3) le conoïde caractéristique ayant pour sommet le point (x, y, z, t) est décrit par l'équation

$$\begin{aligned} & \Gamma(x, y, z, t, \xi, \eta, \zeta, \tau) \\ & \equiv (t - \tau)^2 - (x - \xi)^2 - (y - \eta)^2 \frac{\xi - x + t - \tau}{F(\xi - \tau) - F(x - t)} - (z - \zeta)^2 = 0. \end{aligned} \quad (6)$$

Il en résulte que la solution du problème (3)-(5) existe toujours pour $t > 0$. Elle peut être mise sous la forme

$$\begin{aligned} u(x, y, z, t) = \frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_0^{2\pi} \varphi \Big(\xi, y + \sqrt{(x+t-\xi)[F(\xi) - F(x-t)]} \cos \theta, \\ z + \sqrt{t^2 - (x-\xi)^2} \sin \theta \Big) d\theta, \end{aligned} \quad (7)$$

F désigne dans les formules (6), (7) la fonction primitive de f . Il résulte de cette formule que la solution du problème (3)-(5) au point (x, y, z, t) dépend seulement des valeurs de la fonction sur l'intersection du conoïde caractéristique ayant pour sommet le point en question et de l'hyperplan $\tau = 0$. C'est dire que l'équation (3) n'entraîne jamais de diffusion des ondes.

Remarquons qu'on obtient la formule bien connue de Poisson pour l'équation (2) en posant $f = 1$ dans la formule (7).

3. Il nous reste à montrer que dans le cas général l'équation (3) ne peut pas être réduite à l'équation (2) au moyen d'un changement des variables et de la fonction inconnue. On considère comme toujours [64] les transformations jouissant des propriétés suivantes:

a) transformations $x'^i = x'^i(x)$ des variables indépendantes qui ne sont pas dégénérées;

b) changement de la fonction inconnue $u' = \lambda(x)u$, $\lambda(x) \neq 0$;

c) multiplication de l'équation par une fonction $\mu(x) \neq 0$.

Considérons l'espace de Riemann V_n dont le tenseur métrique $g_{ij}(x)$ est lié au coefficient de l'équation (1) par les relations $g_{ij}g^{jk} = \delta_i^k$. Les applications satisfaisant aux conditions a), b), c) transforment l'espace V_n en un espace qui lui est conforme. L'espace de Riemann lié à l'équation (2) est un espace plan. Donc, l'équation (3) ne peut pas être réduite à l'équation (2) si l'espace V_4 lié à l'équation (3) n'est pas un espace plan-conforme. Dans le cas contraire, l'équation (3) peut être réduite à l'équation (2) [71]. Donc notre problème est réduit à la considération du tenseur de courbure conforme. L'évanouissement de ce dernier est nécessaire et suffisant pour que l'espace soit plan-conforme. En s'appuyant sur cet énoncé on s'aperçoit que la condition $f(\sigma) = (a\sigma + b)^2$, a et b étant des constantes, est la condition nécessaire et suffisante pour la possibilité réduire l'équation (3) à l'équation (2).

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Paper 10

Conformal invariance and Huygens' principle

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I. The wave equation

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0 \quad (1)$$

has the remarkable property of satisfying Huygens' principle. This means that for equation (1) the solution of Cauchy's problem at the point $\mathbf{x} = (x, y, z, t)$ depends only on the values of the initial data in an arbitrarily small neighborhood of the intersection of the characteristic conoid (with its vertex at the point \mathbf{x}) with the initial data surface. Hadamard [61], [62], [64] posed the problem of describing the entire class of linear second-order hyperbolic equations

$$g^{ij}(\mathbf{x})u_{ij} + b^i(\mathbf{x})u_i + c(\mathbf{x})u = 0 \quad (2)$$

satisfying Huygens' principle (see also [63], [35], [150], [40], [11]). We shall consider the case of only four independent variables x^i ($i = 1, \dots, 4$). Thus it will be assumed below that in (2) the sum with respect to the indices i and j is taken from 1 to 4.

Huygens' principle is invariant under the following transformations known as equivalence transformations:

*Editor's note. The present translation incorporates suggestions made by the author.

- a) a non-degenerate change of coordinates: $x'^i = x'^i(\mathbf{x})$;
- b) a linear substitution of the dependent variable: $u' = \lambda(\mathbf{x})u$, $\lambda(\mathbf{x}) \neq 0$;
- c) multiplication of Equation (2) by any function $\nu(\mathbf{x}) \neq 0$.

Therefore two equations of form (2) that can be obtained from one another by such transformations are considered to be equivalent.

Mathisson [128] has proved that any equation (2) with constant coefficients $g^{ij}(i, j = 1, \dots, 4)$ that satisfies Huygens' principle is equivalent to the wave equation (this is not true if there are more than four variables [164]). It would seem that Mathisson's result proved Hadamard's presumption that an equation of type (2) satisfies Huygens' principle only if this equation is equivalent to the wave equation (Hadamard's conjecture). Yet recently we have given an example* of an equation of type (2) that satisfies Huygens' principle without being equivalent to the wave equation [92].

This example was considered in an analysis of equations (2) that have "good" group properties [71]. It turns out that the group properties of equations (2) and the validity of Huygens' principle are closely related. Namely, *if the Riemannian space with a metric tensor $g_{ij}(\mathbf{x})$ has a "nontrivial" (see below) conformal group, then Huygens' principle holds for Equation (2) if and only if this equation is conformally invariant.* This paper contains a basic proof of this assertion. An explicit formula is presented for the solution of Cauchy's problem for any conformally invariant equation of type (2) with nontrivial conformal group. The latter equations comprise, of course, the equation (1) as well as the example presented in [92].

II. A Lie group G is called the *conformal group* of a Riemannian space V_4 with a metric tensor g_{ij} if any one-parameter subgroup of G with an infinitesimal operator

$$X = \xi^i(\mathbf{x}) \frac{\partial}{\partial x^i}$$

satisfies Killing's equations [46]

$$\xi_{i,j} + \xi_{j,i} = \mu(\mathbf{x})g_{ij} \quad (i, j = 1, \dots, 4). \quad (3)$$

Here $\xi_i = g_{ij}\xi^j$ and the subscripts after the comma denote covariant differentiation. If $\mu(\mathbf{x}) \equiv 0$ for all one-parameter subgroups of G then the group G is called the *group of motions*. The conformal group of the space V_4 is said to be *trivial* if it is a group of motions in a certain Riemannian space which is conformal to the space V_4 . The conformal group which is not trivial is called a *nontrivial conformal group* [149]. For simplicity we shall confine ourselves to the case of analytic functions $g_{ij}(\mathbf{x})$.

*After writing this article, the author has learned that a similar example was given earlier by Günther [60].

Below we shall use the following lemma which is a consequence of the results of Biljalov (see, for example, [149], Chapter 7).

Lemma 10.1. Any space V_4 of signature $(- - - +)$ with a nontrivial conformal group can be reduced by a change of coordinates and transition to a conformal space to a space with a tensor g^{ij} ($g_{ij}g^{jk} = \delta_i^k$) of the form

$$g^{ij}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -f(x^1 - x^4) & -h(x^1 - x^4) & 0 \\ 0 & -h(x^1 - x^4) & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f - h^2 > 0. \quad (4)$$

Note that the order of the conformal group in V_4 with a nontrivial conformal group can be equal to 6, 7, or 15.

III. Let us rewrite the equation (2) in the form

$$L(u) \equiv g^{ij}u_{,ij} + a^i u_{,i} + cu = 0 \quad (5)$$

by using covariant derivatives in a Riemannian space V_4 with a metric tensor g_{ij} . The properties of invariance of equation (5) under continuous transformation groups are described in [139] with the aid of the functions

$$K_{ij} = a_{i,j} - a_{j,i} \quad (i, j = 1, \dots, 4)$$

and

$$H = -2c + a^i_{,i} + \frac{1}{2} a^i a_i + \frac{1}{3} R,$$

where R is the scalar curvature of the space V_4 . In fact, the coordinates $\xi^j(\mathbf{x})$ of the infinitesimal operator of any one-parameter subgroup of a Lie group admitted by equation (5) are specified as the solutions of the equations (3) and of the equations

$$\xi^k H_{,k} + \mu H = 0, \quad (K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0 \quad (i, j = 1, \dots, 4). \quad (6)$$

Therefore the group, admitted by equation (5), will be a subgroup of the group of conformal transformations of V_4 .

In the sequel it is important for us to ascertain which equations (5) are invariant under the *entire* conformal group in V_4 (such equations are said to be *conformally invariant*). For spaces with a non-trivial conformal group this problem is solved by the following statement.

Theorem 10.1. In any space V_4 of signature $(- - - +)$ with a nontrivial conformal group, any conformally invariant equation of type (5) is equivalent to the equation

$$g^{ij}u_{,ij} + \frac{1}{6} Ru = 0. \quad (7)$$

Proof. By virtue of Lemma 10.1 it suffices to prove this theorem for spaces with a tensor $g^{ij}(\mathbf{x})$ of the form (4) for which a conformal group is easy to calculate [149]. By solving equations (6), we obtain

$$K_{ij} = 0 \quad (i, j = 1, \dots, 4), \quad H = 0, \quad (8)$$

whence follows the assertion of the theorem (see, e.g. [64] and [139]).

It is worth noting that for spaces with a trivial conformal group there exists, in addition to equation (7), at least one more conformally invariant equation which is not equivalent to (7).

IV. Equation (7) in a space with a tensor g^{ij} of the form (4) is equivalent to the equation

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - 2h(x-t)u_{yz} - u_{zz} = 0. \quad (9)$$

Let us consider for equation (9) the Cauchy problem with initial data

$$u|_{t=0} = 0, \quad (9_1)$$

$$u_t|_{t=0} = \varphi(x, y, z). \quad (9_2)$$

Here $x = x^1$, $y = x^2$, $z = x^3$ and $t = x^4$. The constraint (9₁) is not essential, since for equation (9) it is possible to reduce Cauchy's problem with arbitrary initial data to the problem (9), (9₁), (9₂), this constraint having no effect on the presence or absence of Huygens' principle.

The solution of problem (9), (9₁), (9₂) has the form

$$u(x, y, z, t) = \frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_0^{2\pi} \varphi(\xi, y + A \cos \theta, z + B \cos \theta + C \sin \theta) d\theta, \quad (10)$$

where

$$\begin{aligned} A &= \{(x+t-\xi)[F(\xi) - F(x-t)]\}^{1/2}, \\ B &= [H(\xi) - H(x-t)] \left[\frac{x+t-\xi}{F(\xi) - F(x-t)} \right]^{1/2}, \\ C &= \left\{ (x+t-\xi) \left[\xi - x + t - \frac{[H(\xi) - H(x-t)]^2}{F(\xi) - F(x-t)} \right] \right\}^{1/2}, \end{aligned}$$

F and H being the primitives of the functions f and h .

Formula (10) shows that Huygens' principle holds for equation (9) with any $f, h \in C^1(R)$, $f > h^2$. Indeed, the solution of the problem (9), (9₁), (9₂) at the point \mathbf{x} depends only on the values of the function φ at the intersection of the hyperplane $\tau = 0$ with a characteristic conoid whose

vertex is at the point \mathbf{x} and whose equation is $\Gamma(\mathbf{x}, \boldsymbol{\xi}) = 0$. Here $\Gamma(\mathbf{x}, \boldsymbol{\xi})$ is the square of the geodesic distance between the points $\mathbf{x} = (x, y, z, t)$ and $\boldsymbol{\xi} = (\xi, \eta, \zeta, \tau)$. For equation (9) it has the form

$$\begin{aligned} \Gamma(\mathbf{x}, \boldsymbol{\xi}) = & (t - \tau)^2 - (x - \xi)^2 \\ & - \frac{x - t - \xi + \tau}{(x - t - \xi + \tau)[F(x - t) - F(\xi - \tau)] - [H(x - t) - H(\xi - \tau)]^2} \\ & \times \{ (x - t - \xi + \tau)(y - \eta)^2 - 2[H(x - t) - H(\xi - \tau)](y - \eta)(z - \zeta) \\ & + [F(x - t) - F(\xi - \tau)](z - \zeta)^2 \}. \end{aligned} \quad (11)$$

Our example in [92] corresponds to the case $h \equiv 0$. By setting $h \equiv 0$ and $f \equiv 1$, we obtain from (10) Poisson's well-known formula for the wave equation (1).

V.

Theorem 10.2. Let the Riemannian space V_4 of signature $(- - - +)$ have a nontrivial conformal group. Then equation (5) satisfies Huygens' principle if and only if it is conformally invariant, i.e. it is equivalent to equation (7).

Let us note that this theorem contains the result of Mathisson [128].

Theorem 10.2 can be proved by the method suggested by Hadamard [64] in the case $g^{ij} = \text{const.}$ By virtue of Lemma 10.1 and Theorem 10.1 it suffices to prove that the validity of Huygens' principle for equation (5) with leading coefficients (4) implies the validity of equations (8). We will show that equations (8) hold for the adjoint equation to (5). This will prove the theorem since equation (7) is self-adjoint.

Hadamard showed that Huygens' principle holds for the adjoint equation to equation (5) if and only if the equation $L(W_0) = 0$ holds along the characteristic conoid with vertex at an arbitrary point \mathbf{x}_0 . Here

$$W_0 = \exp \left\{ -\frac{1}{4} \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{L(\Gamma) - c\Gamma - 8}{s} ds \right\},$$

the integral being taken along the geodesic curve connecting the points $\mathbf{x} = (x, y, z, t)$ and $\mathbf{x}_0 = (x_0, y_0, z_0, t)$, $\Gamma = s^2$ is the square of the geodesic distance, and W_0 is regarded as a function of the point \mathbf{x} .

By using Hadamard's criterion, written in the form $L(W_0) = \lambda\Gamma$ with an undetermined (regular) coefficient $\lambda = \lambda(\mathbf{x})$, we find that equation (8) holds at the point \mathbf{x}_0 . Since the point \mathbf{x}_0 is arbitrary, we hence obtain the assertion of the theorem.

VI. In [71], equation (7) is regarded as an equation describing the propagation of light waves in a Riemannian space V_4 , on the basis of the conformal invariance of this equation. Even earlier (and apparently for the first time), equation (7) was considered in [147] likewise from the point of view of conformal invariance. Theorem 10.2 provides a certain physical reason for regarding equation (7) as a wave equation in a Riemannian space (possibly to within an equivalence transformation b) or c) with $\nu = 1/\lambda$). In fact, by considering the radiation problem [35] and using Theorem 10.2, we can see that if a nontrivial conformal group exists in the space V_4 , sharp light signals will be transmitted and can be received as sharp signals if the propagation of light waves in the space V_4 is described by equation (7).

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Paper 11

Huygens' principle

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The history of the principle which we are going to discuss in the present paper begins with the famous memoir of Christiaan Huygens [32]; subsequently this principle was given an adequate mathematical form (see, for example, [17], [61]). For the classical wave equation

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0, \quad (1)$$

which describes propagation of light disturbances, Huygens' principle states that the wave carrying this disturbance does not leave any trace behind (the disturbance is initially limited in space and time). In other words, Huygens' principle for equation (1) means that the solution of the Cauchy problem at any point $P = (x, y, z, t)$ is determined by the Cauchy data given on the intersection of the initial manifold and the characteristic cone with apex at P .

Hadamard [61] has published a profound analysis of the Cauchy problem for linear second-order hyperbolic equations

$$\sum_{i,j=1}^n g^{ij}(x)u_{ij} + \sum_{i=1}^n b^i(x)u_i + c(x)u = 0, \quad (2)$$

which presents the nature of Huygens' principle in a new light. His was the discovery of a remarkable connection between the properties of the elementary solution of equation (2) and the existence of the Huygens principle

for this equation. For equations of the type (2) the Huygens principle is formulated in the same way as for the wave equation, the only difference being that the characteristic cone is replaced by the characteristic conoid. It was proved that for odd n the Huygens principle never holds, whereas for even $n \geq 4$ it is valid if and only if there is no logarithmic term in the elementary solution (the exceptional case $n = 2$ is discussed in [63]).

The Huygens principle holds, in particular, for wave equations with arbitrary even $n \geq 4$ (and for equations which could be obtained from these by means of simple transformations leaving the equation linear and homogeneous). There are many papers (see, for example, [64], [63], [163], [164], [35], [59]) which are devoted to the most important question of existence of other equations of the type (2) for which the Huygens principle is valid. Examples of such equations for even $n \geq 6$ were constructed by K. Stellmacher [163], [164], and the case $n = 4$, which is of special interest because of its physical significance, was for a long time regarded as exceptional. It was only recently that the desired examples were constructed [60] (and independently in [92]).

Construction of these examples can be interpreted as the first step in description of *the whole class of equations of the type (2) for which the Huygens principle holds*. The solution to this problem formulated by Hadamard ([61], § 149) apparently resides in the clarification of the geometrical facts that lie at the basis of Huygens' principle and whose existence is indicated by available results (see the references).

Below one of these geometrical facts is established case $n = 4$. Namely, it is shown that if in the Riemannian space with the metric tensor $g_{ij}(x)$ ($g_{ij}(x)g^{ik}(x) = \delta_i^k$) there exists a nontrivial conformal group, then for equation (2) the Huygens principle holds if and only if this equation is conformally invariant. For all these equations (including the wave equation (1) as well as the examples from [60]) the solution of the Cauchy problem can be obtained explicitly.

§ 1 Conformal invariance

We present equation (2) in the form

$$g^{ij}(x)u_{,ij} + a^i(x)u_{,i} + c(x)u = 0, \quad (3)$$

making use of the covariant derivatives $u_{,ij}$ in the Riemannian space V_n with the metric form

$$ds^2 = g_{ij}(x)dx^i dx^j, \quad (4)$$

which corresponds to equation (3) of [61]; here, as usual, the repeated indices imply summation from 1 to n . Since in what follows we shall be concerned with the case of hyperbolic equations (3) with $n = 4$ independent variables, the form (4) at each point $x = (x^1, \dots, x^4)$ from the considered set of values can be assumed to have the signature $(- - - +)$. For convenience we shall also employ the individual notation for coordinates $x = x^1$, $y = x^2$, $z = x^3$, $t = x^4$. The coefficients of equation (3) will be assumed analytical.

The invariance properties of equations of the type (3) with respect to continuous groups of transformations are described [138] by functions

$$K_{ij} = a_{i,j} - a_{j,i} \quad (i, j = 1, \dots, n),$$

$$H = -2c + a^i_{,i} + \frac{1}{2}a^i a_i + \frac{n-2}{2(n-1)}R$$

(R is the scalar curvature of V_n) and the coordinates $\xi^i(x)$ of infinitesimal operators $X = \xi^i \frac{\partial}{\partial x^i}$ of one-parameter subgroups of the group G of transformations of the space V_n ($n \geq 3$). This means that equation (3) is invariant with respect to a given group if and only if for each one-parameter subgroup of G the Killing equations

$$\xi_{i,j} + \xi_{j,i} = \mu(x)g_{ij} \quad (i, j = 1, \dots, n) \quad (5)$$

and the equations

$$(K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0 \quad (i, j = 1, \dots, n), \quad (6)$$

$$\xi^k H_{,k} + \mu H = 0, \quad (7)$$

hold.

The group G for which the Killing equations (5) are valid is called the group of conformal transformations of the space V_n (in short, the conformal group), and the conformal group with all one-parameter subgroups having the property $\mu(x) \equiv 0$ is called the group of motions. A group which is conformal in V_n is conformal in any space conformal to V_n , but, generally speaking, with a different function $\mu(x)$.

Definition 11.1. A group of conformal transformations of the space V_n is called *trivial* if it coincides with a group of motions in a certain space, conformal to V_n . A conformal group which is not trivial is called *nontrivial*, and the space V_n is called a space with a nontrivial conformal group [149].

A flat space is an example of a space with a nontrivial conformal group. Obviously the property of being a space with nontrivial conformal group

is preserved under transformation to a conformal space. In our case when $n = 4$ there is a complete description of spaces V_4 with a nontrivial conformal group due to R.F. Biljalov (see, for example, [149], Chapter VII). This result can be formulated in the following convenient form.

Lemma 11.1. Any space V_4 of signature $(---+)$ with a nontrivial conformal group G , through a transformation of coordinates and a transformation to a conformal space, can be mapped onto a space with the tensor g^{ij} of the type

$$g^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -f(x-t) & -\varphi(x-t) & 0 \\ 0 & -\varphi(x-t) & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Delta = \varphi^2 - f < 0. \quad (8)$$

The order of the group G can take on the values 6, 7, and (in the case of a conformally flat space) 15.

As follows from (5)-(7), the group with respect to which equation (3) is invariant is a subgroup of the group of conformal transformations of the space V_n . We make the following

Definition 11.2. Equation (3), which is invariant with respect to the general group of conformal transformations of the space V_n , is called a *conformally invariant equation* in V_n .

The above properties of equations (3) as well as the Huygens principle are invariant with respect a) to the non-degenerate coordinate transformations $x'^i = x'^i(x)$, b) to the linear transformation of the function $u' = v(x)u$ ($v(x) \neq 0$), and c) to the multiplication of equation (3) by a function $v(x) \neq 0$ [64], [138]. Two equations obtained from each other by the above transformations will be called equivalent, and all further development is carried out up to these equivalence transformations. In what follows we shall make use of the following theorem partially discussed in [71].

Theorem 11.1. Let the space V_4 of signature $(---+)$ have a nontrivial conformal group, and let equation (3) be conformally invariant. Then this equation is equivalent to the equation

$$g^{ij}(x)u_{ij} + \frac{1}{6} Ru = 0. \quad (9)$$

Proof. According to Lemma 11.1, the above space V_4 can be transformed to a space with tensor g^{ij} of the type (8) by means of transformations a) and b).

Therefore, taking into account that the equations (9) in conformal spaces are equivalent, it is sufficient to prove this theorem for the spaces described in Lemma 11.1. After the coordinates transformation $x^1 = x + t$, $x^2 = y$, $x^3 = z$, $x^4 = x - t$ the following operators enter the conformal group (these are the only operators that will be made use of):

$$X_\alpha = \frac{\partial}{\partial x^\alpha} \quad (\alpha = 1, 2, 3), \quad X_4 = 2x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}.$$

Since, under the conditions of the theorem, equation (3) is conformally invariant, for all these operators equations (6) should hold. We rewrite these equations in the form

$$\xi^l \frac{\partial K_{ij}}{\partial x^l} + K_{lj} \frac{\partial \xi^l}{\partial x^i} + K_{il} \frac{\partial \xi^l}{\partial x^j} = 0 \quad (i, j = 1, \dots, 4). \quad (6')$$

This was done on the basis of the identity

$$\frac{\partial K_{ij}}{\partial x^l} + \frac{\partial K_{jl}}{\partial x^i} + \frac{\partial K_{li}}{\partial x^j} = 0.$$

For the operator X_α ($\alpha = 1, 2, 3$) this equation has the form $\frac{\partial K_{ij}}{\partial x^\alpha} = 0$ ($i, j = 1, \dots, 4$), so that $K_{ij} = K_{ij}(x^4)$. Solving (6') for X_4 , we obtain

$$K_{ij} = 0 \quad (i, j = 1, \dots, 4). \quad (10)$$

Equation (7) can be solved in a similar way. Taking into account that $\mu(x) \equiv 0$ for the operators X_α ($\alpha = 1, 2, 3$), we obtain from (7) that $H = H(x^4)$. Solving this equation for X_4 and taking into consideration that in this case $\mu \neq 0$, we arrive at

$$H = 0. \quad (11)$$

As follows from equation (10) [64], (3) is equivalent to a certain equation (of the same type) with $a^i = 0$ ($i = 1, \dots, 4$) and from (11) and from the definition of H we get $C = R/6$. The theorem is proved.

Note that for spaces with a trivial conformal group there exist at least two conformally invariant equations which are not equivalent to each other. In order to show this, it is sufficient to consider a conformal space in which the conformal group is the group of motions, so that $\mu(x) \equiv 0$, and to introduce two equations with the coefficients $a^i = 0$ ($i = 1, \dots, 4$). In one equation we set $H = 0$, and in the other $H = 1$. These equations are obviously conformally invariant and not equivalent to each other, since during equivalence transformations the quantity H can acquire at most a nonzero factor [138].

§ 2 Geodesic distance

When discussing the Huygens principle for equations (3) in spaces with a nontrivial conformal group, we shall need the square of geodesic distance [61] in spaces V_4 with tensor g^{ij} of the type (8). The geodesic distance can be computed in the following way.

Let a fixed point $P \in V_n$ and a variable point $Q \in V_n$ lie on a geodesic line. On the latter we introduce as a parameter the arc length s read from the point P . Then the coordinates x^i of the point Q will be functions $x^i = x^i(s)$ satisfying the equations

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i(x) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \quad (i = 1, \dots, n) \quad (12)$$

with the initial conditions

$$x^i|_{s=0} = x_0^i, \quad \frac{dx^i}{ds}|_{s=0} = \alpha^i \quad (i = 1, \dots, n), \quad (13)$$

where Γ_{jk}^i are the Christoffel symbols in V_n , x_0^i are the coordinates of the point P , and the constant vector $\alpha = (\alpha^1, \dots, \alpha^n)$ is subject to the condition

$$g_{ij}(x_0) \alpha^i \alpha^j = 1, \quad (14)$$

following from (4). Let

$$x^i = x^i(s, x_0, \alpha) \quad (i = 1, \dots, n) \quad (15)$$

be the solution of problem (12), (13). From (15) we determine the quantities α^i and substitute the result $\alpha^i = \psi^i(s, x_0, x)$ ($i = 1, \dots, n$) in (14). After solving the resulting equation $g_{ij}(x_0) \psi^i(s, x_0, x) \psi^j(s, x_0, x) = 1$ with respect to s , we shall find the square of the geodesic distance $\Gamma = \Gamma(P, Q) = s^2 = [s(x_0, x)]^2$.

Return now to spaces of the type mentioned in Lemma 11.1. In what follows, whenever these spaces are discussed, we shall use the following notation: $P = (\xi, \eta, \zeta, \tau)$, $Q = (x, y, z, t)$, $(\alpha, \beta, \gamma, \delta)$ is the constant vector appearing in (13), F and Φ are the primitives for f and φ , respectively, and $\Delta = \det \|g^{ij}\| = \varphi^2 - f$. In addition, we shall use abbreviations of the type to $f_0 = f(\xi - \tau)$, $f = f(x - t)$, etc. Calculations according to the above method lead to the following result. The solution of the problem (12), (13) has the form

$$x = \xi + \left(\alpha + \beta a - \frac{\alpha - \delta}{2} f_0 a^2 \right) s - \frac{1}{2} a^2 (F - E_0) - ab(\Phi - \Phi_0),$$

$$\begin{aligned}
y &= \eta + a(F - F_0) + b(\Phi - \Phi_0), \\
z &= \zeta + a(\Phi - \Phi_0) + b(\alpha - \delta)s, \\
x - t &= \xi - \tau + (\alpha - \delta)s,
\end{aligned} \tag{16}$$

where

$$a = \frac{\gamma\varphi_0 - \beta}{\Delta_0(\alpha - \delta)}, \quad b = \frac{\beta\varphi_0 - \gamma f_0}{\Delta_0(\alpha - \delta)}.$$

Solution (16) will take the form (15) if one substitutes in the functions $F = F(x - t)$ and $\Phi = \Phi(x - t)$ the value $x - t$ from the last equation (16), and then substitute the resulting value of x in the last equation. From relation (14), which has the form $(\delta - \alpha)(\alpha + \delta + \beta a + \gamma b) = 1$, we obtain

$$\begin{aligned}
\Gamma &= (t - \tau)^2 - (x - \xi)^2 - \frac{x - \xi - t + \tau}{(x - \xi - t + \tau)(F - F_0) - (\Phi - \Phi_0)^2} \\
&\times [(x - \xi - t + \tau)(y - \eta)^2 - 2(\Phi - \Phi_0)(y - \eta)(z - \zeta) + (F - F_0)(z - \zeta)^2]. \tag{17}
\end{aligned}$$

§ 3 The Cauchy problem

According to § 1, any conformally invariant equation (3) in a space V_4 with a nontrivial conformal group is equivalent to the equation

$$\mathbf{L}u \equiv u_{tt} - u_{xx} - f(x - t)u_{yy} - 2\varphi(x - t)u_{yz} - u_{zz} = 0 \tag{18}$$

with certain functions f and φ . This follows from the equivalence of equations (18) and (9) with coefficients of the type (8). Consider a Cauchy problem

$$\mathbf{L}u = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = h(x, y, z), \tag{19}$$

which is obtained from the Cauchy problem

$$\mathbf{L}u = 0, \quad u|_{t=0} = g(x, y, z), \quad u_t|_{t=0} = h(x, y, z), \tag{20}$$

with arbitrary g and h . Indeed, if V is the solution of the problem (19) with $V_t|_{t=0} = g(x, y, z)$ and W the solution of (19) with $W_t|_{t=0} = -h(x, y, z) + g_x(x, y, z)$, then $u = V_t + V_x - W$ is the solution of the problem (20). It is easy to see that this transformation does not affect the Huygens principle.

The solution of the problem (19) can be obtained in the following way. Perform a Fourier transformation with respect to the variables y and z :

$$\hat{u}(x, \lambda, \mu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z, t) e^{-i(\lambda y + \mu z)} dy dz.$$

As a result, (18) takes on the form

$$\hat{u}_{tt} - \hat{u}_{xx} + (f\lambda^2 + 2\varphi\lambda\mu + \mu^2)\hat{u} = 0,$$

which by the change of variables

$$\bar{x} = -\frac{1}{2}\lambda^2 F(x-t) - \lambda\mu\Phi(x-t) - \frac{1}{2}\mu^2(x-t), \quad \bar{t} = \frac{1}{2}(x+t)$$

is transformed to $\hat{u}_{\bar{x}\bar{t}} + \hat{u} = 0$, so that for this equation we have an explicit expression of the Riemann function $R(\xi, \tau; x, t)$. It has the form

$$R = I_0\left(\sqrt{(x-\xi+t-\tau)[(F-F_0)\lambda^2 + 2(\Phi-\Phi_0)\lambda\mu + (\xi-x-\tau+t)\mu^2]}\right)$$

where I_0 is the Bessel function. This allows one to obtain explicitly the solution of the Cauchy problem for the resulting equation. Then, performing the inverse Fourier transformation, we can write the solution of the Cauchy problem (19) in the form

$$u(x, y, z, t) = \frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Ih(\xi, \eta, \zeta) d\eta d\zeta, \quad (21)$$

where

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\xi, 0; x, t) e^{-i[\lambda(y-\eta) + \mu(z-\zeta)]} d\lambda d\mu. \quad (22)$$

From the condition $f - \varphi^2 > 0$ follows validity of the inequality $f\lambda^2 + 2\varphi\lambda\mu + \mu^2 > 0$ for $\lambda^2 + \mu^2 \neq 0$, whence, after integration from $x-t$ to ξ , we obtain that the quadratic form in λ and μ appearing in the expression for $R(\xi, 0; x, t)$ is positive definite. Therefore this form can be presented as a sum of squares, and then one can compute the integral (22). Substituting the value of this integral in (21) and making use of the properties of the special functions, we arrive at the following final version of the solution of the problem (19):

$$u(x, y, z, t) = \frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_0^{2\pi} h(\xi, y + A \cos \theta, z + B \cos \theta + C \sin \theta) d\theta, \quad (23)$$

where

$$A = \sqrt{(x+t-\xi)[F(\xi) - F(x-t)]},$$

$$B = [\Phi(\xi) - \Phi(x-t)] \sqrt{\frac{x+t-\xi}{F(\xi) - F(x-t)}},$$

$$C = \sqrt{(x+t-\xi) \left(\xi - x + t - \frac{[\Phi(\xi) - \Phi(x-t)]^2}{F(\xi) - F(x-t)} \right)}.$$

Direct substitution of expression (23) into (19) shows that this function is indeed a solution of the Cauchy problem (19). We note only that the computations are best performed if one introduces a change of variables $\bar{x} = x + t$, $\bar{t} = x - t$.

One can see from equation (23) that for equation (18), and therefore for any conformally invariant equation (3) in spaces V_4 with a nontrivial conformal group, Huygens' principle is valid.

§ 4 The Huygens principle

The conclusion about the validity of the Huygens principle for equation (18) was made on the basis of the expression for the solution of the Cauchy problem. Instead, one can use the Hadamard criterion [61], [64]. According to Hadamard, the Huygens principle for equation (3) is valid if and only if

$$M(W_0) = 0 \quad \text{for} \quad \Gamma(Q, P) = 0 \quad (24)$$

for arbitrary points $P = (\xi, \eta, \zeta, \tau)$ and $Q = (x, y, z, t)$, where

$$W_0 = \exp \left\{ -\frac{1}{4} \int_P^Q [M(\Gamma) - c\Gamma - 8] \frac{ds}{s} \right\}$$

(the integral is computed along a geodesic connecting the points P and Q), and M is the adjoint operator for (3). This operator is applied to the function W_0 which depends on the point Q . For equation (18) the function W_0 depends only on $x - t$ (as a function of the point Q), so that the validity of (24) is obvious.

Examine now the validity of the Huygens principle for an equation which is obtained from (18) (but not equivalent to it) by addition of lower order terms to the left-hand side. In other words, we want to know whether in spaces V_4 with a nontrivial conformal group there exist nonconformally invariant equations of the type (3) for which the Huygens principle holds. The following theorem gives a negative answer to this question, and thus completely solves the problem of the Huygens principle for spaces V_4 with a nontrivial conformal group.

Theorem 11.2. Let a Riemannian space V_4 of signature $(- - - +)$ have a nontrivial conformal group. Then for equation (3) in this space the Huygens principle holds if and only if this equation is conformally invariant, i.e. equivalent to equation (9).

Proof. As follows from § 1 and § 3, it is sufficient for us to prove that the validity of the Huygens principle for equation (3) with coefficients g^{ij} of the type (8) implies the validity of equations (10) and (11). This can be shown if one applies to our equation the condition (24). But it is sufficient to make use of certain consequences of (24) obtained in [59]. In that paper, among the necessary conditions for the validity of the Huygens principle for equation (3) for $n = 4$, the following relations were obtained:

$$1^\circ. \quad C - \frac{1}{2} a^i_{,i} - \frac{1}{4} a^i a_i = \frac{R}{6},$$

i.e. equation (11).

$$2^\circ. \quad g^{pq}(s_{pij,q} - \frac{1}{2} C^r_{ijq} L_{rp}) = -\frac{5}{4} g^{pq}(K_{pi} K_{qj} - \frac{1}{4} g_{ij} g^{rl} K_{pr} K_{ql}), \quad (25)$$

where

$$L_{rp} = -R_{rp} + \frac{1}{6} g_{rp} R, \quad 2s_{pij} = L_{ij,p} - L_{pj,i},$$

C^r_{ijq} is the tensor of conformal curvature and R_{rp} is the Ricci tensor.

We shall prove the theorem if we demonstrate that (25) implies (10) when the coefficients of the second derivatives in equation (3) have the form (8). We know already that the Huygens principle is valid for equation (18), so that in this case conditions (25) should be satisfied. But for equation (18) the quantities K_{ij} ($i, j = 1, \dots, 4$) are equal to zero, so that the right-hand sides, and therefore the left-hand sides, of equations (25) are also equal to zero. The left-hand sides of (25) depend, however, not on the lower order coefficients of equation (3), but only on its higher order coefficients. Therefore, for the equations which we are concerned with, the left-hand sides of equations (25) are always equal to zero, and in our case (25) takes the form

$$\begin{aligned} & K_{1i} K_{1j} + f K_{2i} K_{2j} + \varphi(K_{2i} K_{3j} + K_{3i} K_{2j}) + K_{3i} K_{3j} - K_{4i} K_{4j} \\ & + \frac{1}{2} g_{ij} [f K_{12}^2 + K_{13}^2 - \Delta K_{23}^2 + 2\varphi K_{12} K_{13} - K_{14}^2 - f K_{24}^2 \\ & - 2\varphi K_{24} K_{34} - K_{34}^2] = 0, \quad i, j = 1, \dots, 4. \end{aligned} \quad (26)$$

Denote the expression in the square brackets by $2N$, and set successively $(i, j) = (1, 1), (2, 2), (2, 3), (3, 3), (4, 4)$. Then we obtain the following system of equations:

$$f K_{12}^2 + 2\varphi + K_{12} K_{13} + K_{13}^2 - K_{14}^2 - N = 0, \quad (27)$$

$$K_{12}^2 + K_{23}^2 - K_{24}^2 + \frac{1}{\Delta} N = 0, \quad (28)$$

$$K_{12}K_{13} - K_{24}K_{34} - \varphi K_{23}^2 - \frac{\varphi}{\Delta}N = 0, \quad (29)$$

$$K_{13}^2 + fK_{23}^2 - K_{34}^2 + \frac{f}{\Delta}N = 0, \quad (30)$$

$$K_{14}^2 + fK_{24}^2 + 2\varphi K_{24}K_{34} + K_{34}^2 + N = 0. \quad (31)$$

Eliminate N from (28) and (29), and then from (28) and (30). This will result in the equations

$$K_{12}K_{13} - K_{24}K_{34} = \varphi(K_{24}^2 - K_{12}^2), \quad (32)$$

$$K_{34}^2 - K_{13}^2 = f(K_{24}^2 - K_{12}^2). \quad (33)$$

Subtracting (27) from (31) and taking (32) and (33) into account, we obtain $K_{14}^2 - \Delta K_{23}^2 = 0$, from which follows

$$K_{14} = 0, \quad K_{23} = 0. \quad (34)$$

By virtue of (32)-(34), equation (27) takes on the form

$$(K_{13} + \varphi K_{12})^2 - \Delta K_{24}^2 = 0,$$

From this equation, as before, we obtain

$$K_{24} = 0, \quad K_{13} = -\varphi K_{12}. \quad (35)$$

Substituting these values of K_{24} and K_{13} in (33), we conclude that all the quantities K_{ij} , ($i, j = 1, \dots, 4$) are equal to zero, which proves our theorem.

In connection with Theorem 11.1 we note an interesting fact. As follows from the examples discussed in [163], [164], for $n > 4$ a theorem which is similar to this one is not valid. Namely, with $n \geq 6$ the equations considered in [163], [164], for which the Huygens principle holds, contain higher order derivatives in the form of the wave operator depending on the respective number of variables. However, in this case also the Huygens principle is related to the invariance properties of the respective equations. When $n = 6$, for example, the equation discussed by K. Stellmacher (see, in particular, [163]) has the form

$$\square u - \frac{\lambda}{(1 - r^2)^2} u = 0,$$

where

$$\square u = u_{nn} - \sum_{i=1}^{n-1} u_{ii}, \quad r^2 = (x^n)^2 - \sum_{i=1}^{n-1} (x^i)^2, \quad \lambda = \text{const.},$$

i.e. it belongs to the class of equations invariant with respect to groups of motions in V_n of the maximal order $\frac{n(n+1)}{2}$ (see [138]). As follows from Stellmacher's results, for $n = 6$ the Huygens principle for the equations of the above type is valid only for $\lambda = 8!$ When dimensions greater than 6 are considered, equations with even weaker properties of invariance appear, for which the Huygens principle holds. This can be demonstrated with certain examples from [164]. Note further that equations of the type (18) for which the Huygens principle is valid could be formulated also for higher dimensions, and new examples could be generated which are different from Stellmacher's. This, however, will not be of great interest, until some general results are obtained.

In conclusion, consider one example of space V_4 with a nontrivial conformal group. Introduce a space with the metric form

$$ds^2 - (1+t)dx^2 - dy^2 - dz^2 + dt^2, \quad t \geq 0. \quad (36)$$

Solving the Killing equation (5), we find that the order of the conformal group for this space is 5, so that we indeed have a space with a trivial conformal group. After some calculations one can see that for this space equations (25) are not compatible (it is sufficient to consider these equations for values $(i, j) = (1, 1), (2, 2), (3, 3), (4, 4)$). Therefore, *in the space V_4 with the metric form (36) the Huygens principle cannot be valid for any equation of the type (3).*

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Paper 12

Conservation laws in hydrodynamics

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Original unabridged version of paper [80].

The conservation of energy, momentum and angular momentum well known for any continuous medium satisfying quite general assumptions [160] arise due to certain symmetry properties of the medium under consideration. For instance, the conservation of angular momentum follows from Boltzman's postulate on symmetry property of the stress tensor. However, to the best of my knowledge, there is no publication where the classical conservation laws are derived from the symmetry properties via Noether's theorem. In consequence, it has not been clarified yet if there are new conservation laws corresponding to additional symmetries of gasdynamic equations found by L.V. Ovsyannikov some ten years ago in [138].

The present paper is aimed at filling this gap. We deal with equations describing the flow of a *perfect polytropic gas* which has more symmetries than equations of motion of an arbitrary gas [138]. Using the extended symmetries we derive the classical conservation laws via Noether's theorem and find new conservation laws other than the classical ones.

§ 1 Ideal compressible fluid

Consider the flow of a perfect polytropic gas described by the equations

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p &= 0, \\ \rho_t + \mathbf{v} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{v} &= 0, \\ p_t + \mathbf{v} \cdot \nabla p + \gamma p \operatorname{div} \mathbf{v} &= 0, \end{aligned} \tag{1}$$

where $\gamma = c_p/c_v = \text{const}$. In what follows, the position vector of a fluid particle with the coordinates x^i is denoted by $\mathbf{r} = (x^1, \dots, x^n)$, where n assumes the values 1, 2, and 3 for one-dimensional, planar and spatial flows, respectively. The dot between the vectors indicates the scalar product of n -dimensional vectors, in particular

$$\mathbf{v} \cdot \nabla = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}.$$

Equations (1) admit the Lie algebra spanned by the operators

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad Y_0 = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad Y_i = t \frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i}, \\ X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j}, \quad (i < j), \quad i, j = 1, \dots, n, \end{aligned} \quad (2)$$

and

$$Z_0 = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}, \quad Z_1 = t \frac{\partial}{\partial t} - v^i \frac{\partial}{\partial v^i} + 2\rho \frac{\partial}{\partial \rho}. \quad (3)$$

The operators (2) generate the classical Galilean group (translations, rotations, similarity and Galilean transformations) admitted by the equations of motion of the general ideal gas:

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p &= 0, \\ \rho_t + \mathbf{v} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{v} &= 0, \\ p_t + \mathbf{v} \cdot \nabla p + A(p, \rho) \operatorname{div} \mathbf{v} &= 0 \end{aligned}$$

with an arbitrary function $A(p, \rho)$. Thus, the group admitted by the equations of motion (1) of the polytropic gas is wider than that of an arbitrary ideal gas and is obtained by augmenting the Galilean group by additional similarity transformations generated by the operators (3) (see [138]).

Moreover, it is shown in [138] that one arrives at a further extension of the symmetry when

$$\gamma = \frac{n+2}{n}. \quad (4)$$

Namely, the maximal Lie algebra admitted by equations (1) with the polytropic exponent (4) is obtained by adding to the basic generators (2)-(3) the following operator (the generator of the projective group):

$$Z_2 = t^2 \frac{\partial}{\partial t} + t x^i \frac{\partial}{\partial x^i} + (x^i - t v^i) \frac{\partial}{\partial v^i} - n t \rho \frac{\partial}{\partial \rho} - (n+2) t p \frac{\partial}{\partial p}. \quad (5)$$

Let us turn now to derivation of the conservation laws associated with the symmetries of equations (1). In order to utilize Noether's theorem, we proceed as follows. First let us consider particular flows of the gas by assuming that the flow is potential,

$$\mathbf{v} = \nabla \Phi, \quad (6)$$

and isentropic, i.e. the entropy $S = \text{const.}$ Then the equation of state of the gas, $p = \sigma(S)\rho^\gamma$, has the form

$$p = C \rho^\gamma, \quad C = \text{const.} \quad (7)$$

Let us set $C = 1$ and use the Cauchy-Lagrange integral written in the form

$$\Phi_t + \frac{1}{2}|\nabla \Phi|^2 + \frac{\gamma}{\gamma-1} \rho^{\gamma-1} = 0, \quad (8)$$

where $|\nabla \Phi|^2 = \sum_{i=1}^n \Phi_i^2$. I use the common notation Φ_t and Φ_i for the partial derivatives of Φ with respect to t and x^i , respectively. In the above assumptions, one can replace the system (1) by the following single non-linear second-order partial differential equation for the potential $\Phi(t, \mathbf{r})$:

$$\Phi_{tt} + 2\nabla \Phi \cdot \nabla \Phi_t + \nabla \Phi \cdot (\nabla \Phi \cdot \nabla) \nabla \Phi + (\gamma-1) \left(\Phi_t + \frac{1}{2}|\nabla \Phi|^2 \right) \nabla^2 \Phi = 0. \quad (9)$$

Indeed, the first equation of the system (1), in view of (6) is written

$$\nabla \Phi_t + (\nabla \Phi \cdot \nabla) \nabla \Phi + \gamma \rho^{\gamma-2} \nabla \rho = 0. \quad (10)$$

Equation (10) is obtained merely by differentiating equation (8) with respect to the variables x^i . Furthermore, the third equation of the system (1) is satisfied identically for isentropic flows; formally it can also be obtained from the second equation of the system (1) by using equation (7). Thus, it remains to rewrite the second equation of the system (1) in terms of the potential Φ . To this end, we differentiate equation (8) with respect to t :

$$\Phi_{tt} + \nabla \Phi \cdot \nabla \Phi_t + \gamma \rho^{\gamma-2} \rho_t = 0, \quad (11)$$

and express $\rho^{\gamma-1}$, $\nabla \rho$ and ρ_t via Φ from the equations (8), (10) and (11), respectively. Now we substitute these expressions for $\rho^{\gamma-1}$, $\nabla \rho$ and ρ_t as well as the equation $\text{div } \mathbf{v} = \nabla^2 \Phi$ into the second equation of the system (1) and arrive at equation (9).

Equation (9) can be obtained from the variational principle with the Lagrangian

$$\mathcal{L} = \left(\Phi_t + \frac{1}{2}|\nabla \Phi|^2 \right)^{\gamma/(\gamma-1)}.$$

Indeed, one can readily verify that the variational derivative of \mathcal{L} ,

$$\frac{\delta \mathcal{L}}{\delta \Phi} = \frac{\partial \mathcal{L}}{\partial \Phi} - D_t \left(\frac{\partial \mathcal{L}}{\partial \Phi_t} \right) - D_i \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} \right),$$

has the form

$$\frac{\delta \mathcal{L}}{\delta \Phi} = -\frac{\gamma}{(\gamma-2)^2} \left(\Phi_t + \frac{1}{2} |\nabla \Phi|^2 \right)^{(2-\gamma)/(\gamma-1)} \Psi,$$

where Ψ is the left-hand side of equation (9). Hence, equation (9) is equivalent to the Euler-Lagrange equation $\frac{\delta \mathcal{L}}{\delta \Phi} = 0$. Note that D_t and D_i denote the total differentiations in t and x^i , respectively:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + \Psi_t \frac{\partial}{\partial \Psi} + \Psi_{tt} \frac{\partial}{\partial \Psi_t} + \Psi_{it} \frac{\partial}{\partial \Psi_i}, \\ D_i &= \frac{\partial}{\partial x^i} + \Psi_i \frac{\partial}{\partial \Psi} + \Psi_{it} \frac{\partial}{\partial \Psi_t} + \Psi_{ij} \frac{\partial}{\partial \Psi_j}. \end{aligned}$$

Calculating the symmetries of equation (9) I have found that equation (9) with an arbitrary polytropic exponent γ admits the group with the following basic generators:

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \\ Y_0 &= t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} + \Phi \frac{\partial}{\partial \Phi}, \quad Y_i = t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial \Phi} \end{aligned} \quad (12)$$

and

$$\tilde{Z}_0 = \frac{\partial}{\partial \Phi}, \quad Z_1 = t \frac{\partial}{\partial t} - \Phi \frac{\partial}{\partial \Phi}. \quad (13)$$

Furthermore, if the polytropic exponent has the form (4), the admitted Lie algebra is extended. Namely, the following generator is added to (12)-(13):

$$Z_2 = t^2 \frac{\partial}{\partial t} + t x^i \frac{\partial}{\partial x^i} + \frac{1}{2} r^2 \frac{\partial}{\partial \Phi}, \quad (14)$$

where $r^2 = (x^1)^2 + \dots + (x^n)^2$.

An appropriate extension of the operators (12)-(14) to the variables \mathbf{v} , ρ and p yields all the operators (2), (3) and (5) except Z_0 . Namely, we extend the operators (12)-(14) to the derivatives Φ_i and Φ_t by means of the usual prolongation procedure. In the resulting operators, we replace Φ_i by v^i according to the equations (6). The extension to the variables ρ and p is

obtained from the requirement that the extended operators should leave invariant the equation (8) written in the form

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} = 0. \quad (15)$$

Consider, e.g. the operator Z_1 from (13). Taking its prolongation to Φ_t, Φ_i and extension to ρ and p with undetermined coefficients η, ζ :

$$\widehat{Z}_1 = t \frac{\partial}{\partial t} - \Phi \frac{\partial}{\partial \Phi} - 2\Phi_t \frac{\partial}{\partial \Phi_t} - \Phi_i \frac{\partial}{\partial \Phi_i} + \eta \frac{\partial}{\partial \rho} + \zeta \frac{\partial}{\partial p}$$

and writing the invariance condition of equation (15),

$$\widehat{Z}_1 \left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} \right) \Big|_{(15)} = 0,$$

we arrive at the equation

$$\zeta - \frac{p}{\rho} \eta + 2p = 0.$$

We solve it by letting $\eta = 2\rho$, $\zeta = 0$. In \widehat{Z}_1 , we omit the terms with Φ, Φ_t , replace Φ_i by v^i and obtain Z_1 from (3). This procedure leads to the operators (2), (3) and (5) exclusive of Z_0 . Hence, equations (1) and (9) have identical symmetry properties if we associate the operator Z_0 with \widetilde{Z}_0 .

Now we can obtain the conservation laws for equation (9) by using the symmetries (12)-(14) and applying the Noether theorem [133] (a detailed discussion of the Noether theorem and its applications is available in [79]). The conservation laws will be written in the form

$$D_t(A^0) + \sum_{i=1}^n D_i(A^i) = 0. \quad (16)$$

We rewrite the resulting conservation laws in the variables \mathbf{v}, p, ρ using the definition of the potential (6), the connection (7) between pressure p and density ρ in the isentropic gas flow, and the Cauchy-Lagrange integral (8).

The following notation is used below. An arbitrary n -dimensional volume moving together with the fluid and its boundary are denoted by $\Omega(t)$ and $S(t)$, respectively. The unit outer normal to $S(t)$ is $\boldsymbol{\nu}$. The volume and surface elements are denoted by $d\omega$ and dS , respectively. Furthermore, it is convenient in fluid dynamics to write conservation laws in the form

$$D_t(\tau) + \operatorname{div}(\tau \mathbf{v} + \boldsymbol{\lambda}) = 0. \quad (17)$$

Then the standard procedure (integration over an $(n+1)$ -dimensional tube domain $\Omega \times [t_1, t_2]$ and the divergence theorem) allows one to rewrite the differential conservation law (17) in the integral form

$$\frac{d}{dt} \int_{\Omega(t)} \tau dx = - \int_{S(t)} (\boldsymbol{\lambda} \cdot \boldsymbol{\nu}) dS. \quad (18)$$

Equation (18) is convenient for physical interpretation of conservation laws.

To illustrate the procedure, let us find the conservation law obtained by using the symmetry \tilde{Z}_0 from (13). The general formula for conservation laws* yields the conservation law (16) with the following components:

$$A^0 = \frac{\partial \mathcal{L}}{\partial \Phi_t} = \frac{\gamma}{\gamma-1} \left(\Phi_t + \frac{1}{2} |\nabla \Phi|^2 \right)^{\frac{1}{\gamma-1}},$$

$$A^i = \frac{\partial \mathcal{L}}{\partial \Phi_i} = \frac{\gamma}{\gamma-1} \left(\Phi_t + \frac{1}{2} |\nabla \Phi|^2 \right)^{\frac{1}{\gamma-1}} \Phi_i.$$

Invoking the equations (8) and (6), we have

$$A^0 = \frac{\gamma}{\gamma-1} \left(\frac{\gamma}{1-\gamma} \rho^{\gamma-1} \right)^{\frac{1}{\gamma-1}} = k \rho,$$

$$A^i = \frac{\gamma}{\gamma-1} \left(\frac{\gamma}{1-\gamma} \rho^{\gamma-1} \right)^{\frac{1}{\gamma-1}} v^i = k \rho v^i.$$

Hence, ignoring the immaterial constant factor

$$k = - \left(\frac{\gamma}{1-\gamma} \right)^{\gamma/(\gamma-1)},$$

we obtain the conservation law (17) with $\tau = \rho$ and $\boldsymbol{\lambda} = 0$,

$$D_t(\rho) + \operatorname{div}(\rho \mathbf{v}) = 0.$$

Rewriting it in the integral form (18),

$$\frac{d}{dt} \int_{\Omega(t)} \rho d\omega = 0,$$

we see that its physical meaning is the conservation of mass.

* *Author's note to this 2006 edition:* See Paper 8 in this volume, equation (10).

The conservation laws obtained by applying the above technique to the operators (12)–(14) are summarized below.

The invariance of equation (9) under the one-parameter transformation group $\Phi \rightarrow \Phi + a$ with the parameter a , and the Galilean group (translation in time $t \rightarrow t + a$, and coordinates $x^i \rightarrow x^i + a^i$, rotations, transition to a uniformly moving coordinate system) leads to the following classical conservation laws. They are also valid in a more general case of motion of a continuous medium.

Conservation of mass:

$$\frac{d}{dt} \int_{\Omega(t)} \rho d\omega = 0. \quad (19)$$

Conservation of energy:

$$\frac{d}{dt} \int_{\Omega(t)} \left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{p}{\gamma - 1} \right) d\omega = - \int_{S(t)} p \mathbf{v} \cdot \boldsymbol{\nu} dS. \quad (20)$$

Conservation of momentum:

$$\frac{d}{dt} \int_{\Omega(t)} \rho \mathbf{v} d\omega = - \int_{S(t)} p \boldsymbol{\nu} dS. \quad (21)$$

Conservation of angular momentum:

$$\frac{d}{dt} \int_{\Omega(t)} \rho (\mathbf{r} \times \mathbf{v}) d\omega = - \int_{S(t)} p (\mathbf{r} \times \boldsymbol{\nu}) dS. \quad (22)$$

Center-of-mass theorem:

$$\frac{d}{dt} \int_{\Omega(t)} \rho (t\mathbf{v} - \mathbf{r}) d\omega = - \int_{S(t)} tp \boldsymbol{\nu} dS. \quad (23)$$

The following specific conservation laws hold for a polytropic gas satisfying the condition (4):

$$\frac{d}{dt} \int_{\Omega(t)} [t(\rho |\mathbf{v}|^2 + np) - \rho \mathbf{r} \cdot \mathbf{v}] d\omega = - \int_{S(t)} p(2t\mathbf{v} - \mathbf{r}) \cdot \boldsymbol{\nu} dS, \quad (24)$$

$$\frac{d}{dt} \int_{\Omega(t)} [t^2(\rho |\mathbf{v}|^2 + np) - \rho \mathbf{r} \cdot (2t\mathbf{v} - \mathbf{r})] d\omega = - \int_{S(t)} 2tp(t\mathbf{v} - \mathbf{r}) \cdot \boldsymbol{\nu} dS. \quad (25)$$

The conservation laws (19)–(25) hold for any solutions of equations (1). Furthermore, the following conservation law:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega(t)} \left[\frac{2\gamma + n(\gamma - 1)}{\gamma + 1} t \left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{p}{\gamma - 1} \right) - \rho \mathbf{r} \cdot \mathbf{v} - \frac{n(\gamma - 1) - 2}{\gamma + 1} \rho \Phi \right] d\omega \\ = - \int_{S(t)} p \left(\frac{2\gamma + n(\gamma - 1)}{\gamma + 1} t \mathbf{v} - \mathbf{r} \right) \cdot \boldsymbol{\nu} dS \end{aligned} \quad (26)$$

holds for a potential flow of a polytropic gas with arbitrary γ . If the condition (4) is satisfied, equation (26) is identical with (24).

For spatial flows ($n = 3$), the condition (4) ($\gamma = 5/3$) is satisfied for monatomic gases. Thus, additional conservation laws (24), (25) are valid for monatomic gases.

Similar occurrence of additional conservation laws can be observed also in classical mechanics (motion of a particle in the Newtonian gravitational field or in the Coulomb electrostatic field) as well as in quantum mechanics (the hydrogen atom).

§ 2 Ideal incompressible fluid

Equations describing motion of an ideal incompressible fluid

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0, \quad \operatorname{div} \mathbf{v} = 0 \quad (27)$$

also have additional conservation laws different from the classical conservation laws (19)–(23).

One of these additional conservation laws holds for potential flows and has the form

$$\frac{d}{dt} \int_{\Omega(t)} \left(\frac{n+2}{2} t |\mathbf{v}|^2 - \mathbf{r} \cdot \mathbf{v} - n \Phi \right) d\omega = - \int_{S(t)} p [(n+2) t \mathbf{v} - \mathbf{r}] \cdot \boldsymbol{\nu} dS. \quad (28)$$

Formally the conservation equation (28) can be obtained from equation (26) by letting $\gamma \rightarrow \infty$.

Note that the equation

$$\int_{\Omega} \mathbf{v} d\omega = \int_S (\mathbf{v} \cdot \boldsymbol{\nu}) \mathbf{r} dS$$

holds for any vector \mathbf{v} satisfying the condition $\operatorname{div} \mathbf{v} = 0$. Therefore, the conservation of momentum can be generalized for an ideal incompressible fluid as follows:

$$\frac{d}{dt} \int_{\Omega(t)} \mathbf{f} \cdot \mathbf{v} d\omega = - \int_{S(t)} [p \mathbf{f} \cdot \boldsymbol{\nu} - (\frac{d\mathbf{f}}{dt} \cdot \mathbf{r})(\mathbf{v} \cdot \boldsymbol{\nu})] dS.$$

Here $\mathbf{f} = (f^1(t), \dots, f^n(t))$ is an arbitrary smooth vector-function of time t . This conservation law results from the invariance of equations (27) under the translational movement of the coordinate system, $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{f}(t)$.

Provided that the condition (4) is satisfied, one can treat the equations of a planar ($n = 2$) flow of a polytropic gas as equations of motion of an incompressible fluid in the theory of shallow water [166]. To this end one has to set $p = \frac{1}{2}\rho^2$ in equations (1) and replace the density ρ by gh , where g is the acceleration of gravity and h is the depth of the flow. As a result one arrives at the equations

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + g \nabla h = 0,$$

$$h_t + \mathbf{v} \cdot \nabla h + h \operatorname{div} \mathbf{v} = 0.$$

Furthermore, the special conservation laws (24) and (25) take the form

$$\frac{d}{dt} \int_{\Omega(t)} h[t(|\mathbf{v}|^2 + gh) - \mathbf{r} \cdot \mathbf{v}] d\omega = - \int_{S(t)} \frac{1}{2} gh^2 (2t\mathbf{v} - \mathbf{r}) \cdot \boldsymbol{\nu} dS, \quad (29)$$

and

$$\frac{d}{dt} \int_{\Omega(t)} h[t^2(|\mathbf{v}|^2 + gh) - \mathbf{r} \cdot (2t\mathbf{v} - \mathbf{r})] d\omega = - \int_{S(t)} gth^2 (t\mathbf{v} - \mathbf{r}) \cdot \boldsymbol{\nu} dS, \quad (30)$$

respectively.

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Paper 13

Invariance and conservation laws of continuum mechanics

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Abstract. The object of the present lecture is the discussion of the group-theoretical nature of the conservation theorems for differential equations. To illustrate the use of the general conservation theorems the differential equations of mechanics are chosen.

§ 1 Introduction

In classical mechanics it has been known for a long time that the existence of conservation laws for equations of motion is connected with symmetry properties of mechanical system [99]. It has been shown by Klein [101] and Noether [133] that if the differential equations are derivable from the variational principle, a general and regular procedure for establishment of the conservation laws can be developed, due to the investigation of invariance properties of the variational integral. Since the differential equations of Mathematical Physics are derivable from the variational principle, the Noether’s theorem occurs to be a convenient method allowing to study conservation laws systematically.

The Noether theorem which has a rather general character does not however clarify fully enough a group-theoretical nature of conservation laws. This theorem allows to find several conservation laws of a certain type due to invariance properties of the variational integral. But in the general case

it does not follow from the existence of the conservation laws that the variational integral is invariant.

The general conservation theorem has been recently proved which establishes the full correspondence between invariance properties of the variational problem and existence of conservation laws [75], [79]. Besides, it has been proved that if the variational integral is non-degenerate, this theorem is equivalent to the Noether's theorem. The given lecture contains these results. To illustrate the general conservation theorems the several equations of mechanics are chosen. Other interesting results on the conservation theorems can be found in [20], [29], [43]. The basic conceptions of the group analysis of differential equations [138] which are used later, can be found in Ovsyannikov's lecture [143].

§ 2 Preliminaries

The independent variables of the system of partial differential equations under discussion will be designated as $x = (x^1, \dots, x^n)$, while the dependent variables will be designated as $u = (u^1, \dots, u^m)$. The partial derivatives of the functions with respect to the independent variables will be indicated by the index notation. The derivative of the mapping $u : R^n \rightarrow R^m$ is the $m \times n$ matrix $\partial u = \|u_j^k\|$.

For a system S of partial differential equations we have a conservation law, if n -dimensional vector A with the components $A^i(x, u, \partial u, \dots)$ exists for which the equation

$$\frac{\partial}{\partial x^i} A^i(x, \varphi(x), \partial \varphi(x), \dots) = 0$$

is valid for every solution $u = \varphi(x)$ of the system S . In this equation and the later formulae we shall employ the usual dummy index notation to indicate summations over the coordinates of independent or dependent variables, when the corresponding indices are repeated in a given member. Due to the symbol

$$D_i = \frac{\partial}{\partial x^i} + u_i^k \frac{\partial}{\partial u^k} + u_{ij}^k \frac{\partial}{\partial u_j^k}$$

of partial differentiation over independent variables when variables u^k and their derivatives have been supposed to have been substituted as functions of the independent variables, the conservation equation will be written in the form

$$D_i A^i|_S = 0. \quad (2.1)$$

Vector $A = (A^1, \dots, A^n)$ satisfying the conservation Equation (2.1) will be henceforth called C -vector or, if $n = 1$, C -quantity.

It will be assumed that the system S is derivable by the application of the variational procedure to the variational integral

$$\int \mathcal{L}(x, u, \partial u) dx \quad (2.2)$$

so the system of differential equations under discussion is the Euler equations

$$D_i \frac{\partial \mathcal{L}}{\partial u_i^k} - \frac{\partial \mathcal{L}}{\partial u^k} = 0, \quad k = 1, \dots, m. \quad (2.3)$$

The integral (2.2) is to be extended over an arbitrary region of the space of the independent variables, in which dx represents the volume element. The set of the extremals of the variational integral (2.2), i.e. the solutions of the Euler equations (2.3), will be designated as E .

Let G be a local one-parameter group of transformations

$$y' = F(y, a) \quad (2.4)$$

in R^N with the tangent vector

$$\xi(y) = \left. \frac{\partial F(y, a)}{\partial a} \right|_{a=0}. \quad (2.5)$$

The manifold $M \subset R^N$, which is given by the equation

$$\psi(y) = 0 \quad \text{where} \quad \psi : R^N \rightarrow R^p \quad (p < N), \quad \text{rank } \partial \psi|_M = p$$

is invariant with respect to the group G if and only if

$$\partial \psi \cdot \xi|_M = 0. \quad (2.6)$$

We shall deal with the transformations of the independent variables $x \in R^n$ and dependent variables $u \in R^m$, so $y = (x, u)$ and the transformations (2.4) will be of the form

$$\begin{aligned} x' &= x(x, u, a), \\ u' &= g(x, u, a). \end{aligned} \quad (2.7)$$

In this case the tangent vector (2.5) will be written in the form

$$(\xi, \eta) \quad \text{where} \quad \xi = (\xi^1, \dots, \xi^n), \quad \eta = (\eta^1, \dots, \eta^m). \quad (2.8)$$

Let the function $u = \varphi(x)$ be transformed to the function $u' = \varphi'(x')$ by the transformation (2.7) and the region $\Omega \subset R^n$ be transformed to the region $\Omega' \subset R^n$. Let the set Φ of functions $\varphi : R^n \rightarrow R^m$ be invariant under the transformations (2.7), i. e. $\varphi' \in \Phi$ for any $\varphi \in \Phi$.

Definition 13.1. If

$$\int_{\Omega'} \mathcal{L}(x', \varphi', \partial\varphi') dx' = \int_{\Omega} \mathcal{L}(x, \varphi, \partial\varphi) dx \quad (2.9)$$

for any $\varphi \in \Phi$, any region $\Omega \subset R^n$ and for all values of the group parameter a , then the variational integral (2.2) is called Φ -invariant. If Φ is the set of all (smooth) functions, then the variational integral is called *invariant*.

Remark 13.1. The case of E -invariance is of particular importance to us, i.e. the case when the stationary values of the variational integral are invariant under the transformations (2.7) of the group G .

Definition 13.2. The variational integral (2.2) is called *non-degenerate* if all equations of the system (2.3) have the second order.

Remark 13.2. The Euler equations (2.3) is the system of quasi-linear equations of the form

$$a^{ij}(x, u, \partial u) u_{ij} + b(x, u, \partial u) = 0$$

where a^{ij} , $i, j = 1, \dots, n$, are $m \times m$ matrices and $b = (b^1, \dots, b^m)$. The non-degeneration condition means that

$$\text{rank } \|a^{11} \dots a^{ij} \dots a^{nn}\| = m. \quad (2.10)$$

§ 3 Noether theorem

Theorem 13.1. [133]. If the variational integral (2.2) is invariant with respect to one-parameter group G with the tangent vector (2.8), then the conservation equation

$$D_i A^i = 0, \quad (3.1)$$

where

$$A^i = (\eta^k - u_j^k \xi^j) \frac{\partial \mathcal{L}}{\partial u_i^k} + \mathcal{L} \xi^i, \quad i = 1, \dots, n,$$

is valid for the Euler equations (2.3).

The application of this theorem to the case of invariance of the integral (2.2) with respect to the r -parameter group G_r gives r independent conservation laws. The independence of conservation laws denotes linear independence of corresponding C -vectors. These independent C -vectors A_α , $\alpha = 1, \dots, r$, can be derived by the formula (3.1) applied to r linearly independent tangent vectors $(\xi_\alpha, \eta_\alpha)$, $\alpha = 1, \dots, r$, of the group G_r .

§ 4 Basic conservation theorem

Theorem 13.2. [75]. Let the Euler equations (2.3) admit the group G of transformations (2.7). Then the conservation equation (3.1) is valid if and only if the variational integral (2.2) is E -invariant with respect to the group G .

Proof. Equation (2.9), taking into account that the region Ω is an arbitrary one, can be rewritten in the form

$$\mathcal{L}(x, \varphi, \partial\varphi) = \mathcal{L}(x', \varphi', \partial\varphi')J \quad (4.1)$$

where J is the Jacobian of the transformation $x' = f(x, \varphi(x), a)$. Therefore the condition of E -invariance of the integral (2.2) denotes that Equation (4.1) is valid for any solution $\varphi(x)$ of the Euler equations (2.3).

Now let us consider the group \tilde{G} of transformations in the space of the variables $(x, u, \partial u, \partial^2 u, \Lambda)$, which consists of the second order extension of the group G and the transformation

$$\Lambda' = \Lambda J^{-1} \quad (4.2)$$

of the variable Λ . Then Equation (4.1) is valid for any $\varphi \in E$ if and only if the manifold given by Equations (2.3) and

$$\Lambda = \mathcal{L}(x, u, \partial u) \quad (4.3)$$

is invariant with respect to the group \tilde{G} . Now it is necessary to write down the infinitesimal criterion (2.6) of the invariance of this manifold. Taking into account that Equations (2.3) do not contain the variable Λ and admit the group G , and using the well known extension formula of the tangent vector (2.8) [138], we shall get the criterion of E -invariance of the variational integral (2.2) in the form

$$\xi^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta^k \frac{\partial \mathcal{L}}{\partial u^k} + (D_i \eta^k - u_j^k D_i \xi^j) \frac{\partial \mathcal{L}}{\partial u_i^k} + \mathcal{L} D_i \xi^i = 0 \quad (4.4)$$

for any $\varphi \in E$.

It's easy to verify that the left side of Equation (4.4) is equal to

$$(\eta^k - u_j^k \xi^j) \left(D_i \frac{\partial \mathcal{L}}{\partial u_i^k} - \frac{\partial \mathcal{L}}{\partial u^k} \right) + D_i A^i \quad (4.5)$$

where A^i are defined by (3.1). Therefore E -invariance of the variational integral is equivalent to the conservation equation (3.1).

Remark 13.3. Noether's theorem (Theorem 13.1) is the particular case of Theorem 13.2. Indeed, if the integral (2.2) is invariant with respect to the group G , the Euler equations (2.3) admit the group G . Besides, it is clear that the invariant variational integral occurs to be E -invariant.

Remark 13.4. In Theorem 13.2 linear independence of the C -vectors which are defined in (3.1), is not guaranteed. Nevertheless, all conservation laws which one obtains from Theorem 13.1, may be obtained from Theorem 13.2. Therefore, the latter gives no less independent C -vectors than the first one.

§ 5 Conservation theorem in non-degenerate cases

Theorem 13.3. [75]. Let the variational integral (2.2) be non-degenerate and Euler equations (2.3) admit the group G of transformations (2.7). Then the conservation equation (3.1) is valid if and only if the integral (2.2) is invariant with respect to the group G .

To prove this theorem it is enough to show that if the non-degenerate variational integral is E -invariant with respect to the group G this integral is invariant with respect to the same group. This property of the variational integral follows from the way of Theorem 13.2 proof. Details of the proof are neglected.

Remark 13.5. The simple generalization of the Noether theorem is known, being obtained due to the usage of the so-called divergence transformations [20]. They say, the transformations (2.7) are divergence ones, if for the tangent vector (2.8) the equation

$$\xi^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta^k \frac{\partial \mathcal{L}}{\partial u^k} + (D_i \eta^k - u_j^k D_i \xi^j) \frac{\partial \mathcal{L}}{\partial u_i^k} + \mathcal{L} D_i \xi^i = D_i B^i \quad (5.1)$$

with some vector $B(x, u, \partial u)$ is valid. If the right side of Equation (5.1) does not vanish, the variational integral (2.2) will not be invariant under transformations (2.7), so the conservation equation (3.1) is not valid. However, the conservation equation (2.1) is valid for the vector

$$C = A - B, \quad (5.2)$$

where the vector A is defined by the formula (3.1) and B by (5.1).

Now let's turn to the applications of the general conservation theorems. In Sections § 6, § 7 the cases of non-degenerate variational integrals are considered. In Section § 8 we deal with the case of degenerate variational integral, when far from all the conservation laws of the type (3.1) can be obtained due to Theorem 13.1. It will be shown in Section § 9 that the Noether theorem may be useful in the cases when the equations under discussion are not of the form (2.3).

§ 6 Motion of a free particle in the de Sitter space

A free particle in 4-dimensional space-time V_4 moves along the geodesic. Let $x^i = x^i(\sigma)$, $i = 1, \dots, 4$, be a curve in the space-time V_4 with the fundamental form

$$ds^2 = g_{ij}(x)dx^i dx^j$$

and

$$\dot{x}^i = \frac{dx^i}{d\sigma}, \quad i = 1, \dots, 4.$$

The equations of the particle motion are derivable from the variational principle with the Lagrangian function

$$\mathcal{L} = -mc\sqrt{g_{ij}(x)\dot{x}^i \dot{x}^j} \quad (6.1)$$

where c is the velocity of light and m is the mass of the particle. The parameter σ is the independent variable and the coordinates of the 4-vector $x = (x^1, \dots, x^4)$ are dependent ones. The variational integral (2.2) with the Lagrangian function (6.1) is invariant with respect to the group of motions in V_4 . The tangent vector (2.8) for this group is of the form

$$(0, \eta) \quad (6.2)$$

where $\eta = (\eta^1, \dots, \eta^4)$ is the solution of the Killing equations

$$\eta_{i,j} + \eta_{j,i} = 0, \quad i, j = 1, \dots, 4.$$

The comma in this formula denotes covariant differentiation. Let's choose the length s of a curve as its parameter σ . Then we shall obtain the C -quantity in the form

$$A = mcg_{ij}(x)\dot{x}^i \eta^j \quad (6.3)$$

for every solution η of the Killing equations by the formula (3.1).

We shall consider the space-time V_4 of constant curvature k , known as the de Sitter space. The group of motions in the de Sitter space (the de Sitter group) occurs to be 10-parameter group, so 10 independent conservation laws exist for free motion of a particle in the de Sitter space, as in relativistic mechanics. Below we shall write out all the C -quantities, with the index k indicating these conservation laws corresponding to the motion in the de Sitter space with the curvature k . In particular, C -quantities in relativistic mechanics will have the index 0.

The fundamental form of the de Sitter space can be written in the canonical form

$$ds^2 = \theta^{-2}(c^2 dt^2 - dx^2 - dy^2 - dz^2) \quad (6.4)$$

where

$$\theta = 1 + \frac{k}{4}(c^2 t^2 - x^2 - y^2 - z^2).$$

Below the notations $x = x^1$, $y = x^2$, $z = x^3$, $t = x^4$ will be used. The Greek indexes λ, μ, ν will be changed from 1 to 3, and Latin indexes i, j from 1 to 4. Vector notations will be used for spatial vectors. So, $\mathbf{x} = (x^1, x^2, x^3)$, $\mathbf{x} \cdot \mathbf{v} = \sum_{\lambda=1}^3 x^\lambda v^\lambda$, and $\mathbf{x} \times \mathbf{v}$ is the vector product: $(\mathbf{x} \times \mathbf{v})^\lambda = e_{\lambda\mu\nu} x^\mu v^\nu$, where $e_{\lambda\mu\nu}$ is the usual permutation symbol.

The components $\dot{x}^i = \frac{dx^i}{ds}$ of the 4-velocity in the de Sitter space and the components $v^\lambda = \frac{dx^\lambda}{dt}$ of the physical velocity \mathbf{v} are connected by

$$\dot{x}^\lambda = \frac{\theta v^\lambda}{\sqrt{1 - \beta^2}}, \quad \dot{x}^4 = \frac{\theta}{c\sqrt{1 - \beta^2}}, \quad \text{where } \beta^2 = \frac{|\mathbf{v}|^2}{c^2}. \quad (6.5)$$

Now let's consider the different subgroups of the de Sitter group together with the corresponding C -quantities.

1°. *Conservation of momentum.* In classical and relativistic mechanics the conservation of momentum arises due to the invariance of mechanical system under the translations of the axes. In the de Sitter space, instead of the translation group, we have the 3-parameter group, generated by the tangent vectors of the form (6.2) with

$$\eta_\lambda^i = (\theta - 2)\delta^{\lambda i} + \frac{k}{2}x^\lambda x^i. \quad (6.6)$$

Substituting (6.6) into the formula (6.3) and using the formulae (6.4) and (6.5), we obtain the C -quantities

$$A_\lambda = \frac{m}{\theta\sqrt{1 - \beta^2}} \left\{ (2 - \theta)v^\lambda + \frac{k}{2}(c^2 t^2 - \mathbf{x} \cdot \mathbf{v})x^\lambda \right\}.$$

So, the momentum of a free particle in the de Sitter space is defined by the formula

$$\mathbf{p}_k = \frac{m}{\sqrt{1-\beta^2}} \left\{ \left(\frac{2}{\theta} - 1 \right) \mathbf{v} + \frac{k}{2\theta} (c^2 t^2 - \mathbf{x} \cdot \mathbf{v}) \mathbf{x} \right\}. \quad (6.7)$$

If $k = 0$, this formula gives the well-known formula of the momentum in relativistic mechanics:

$$\mathbf{p}_0 = \frac{m\mathbf{v}}{\sqrt{1-\beta^2}}.$$

2°. *Conservation of energy.* Conservation of energy is the result of the variational integral invariance under the one-parameter subgroup of the de Sitter group generated by the tangent vector (6.2) with

$$\eta^\lambda = \frac{k}{2} x^4 x^\lambda, \quad \eta^4 = \frac{1}{c^2} \left(\theta + \frac{k}{2} |\mathbf{x}|^2 \right).$$

The C -quantity (6.3) in this case is equal to

$$A = \frac{m}{\sqrt{1-\beta^2}} \left\{ 1 + \frac{k}{2\theta} (\mathbf{x} - t\mathbf{v}) \cdot \mathbf{x} \right\}.$$

This quantity multiplied to c^2 is the energy of a free particle in the de Sitter space:

$$\mathcal{E}_k = \frac{mc^2}{\sqrt{1-\beta^2}} \left\{ 1 + \frac{k}{2\theta} (\mathbf{x} - t\mathbf{v}) \cdot \mathbf{x} \right\}. \quad (6.8)$$

For $k = 0$ we have the formula of the relativistic energy,

$$\mathcal{E}_0 = \frac{mc^2}{\sqrt{1-\beta^2}}.$$

3°. *Conservation of angular momentum.* The de Sitter group contains the subgroup of rotations of the axes and the subgroup of the Lorentz transformations. The conservation of angular momentum corresponds to the invariance of the variational integral with respect to the rotations with tangent vectors of the type (6.2), where

$$\eta_\lambda^\mu = e_{\lambda\mu\nu} x^\nu, \quad \eta_\lambda^4 = 0.$$

The expression for angular momentum according to (6.3), (6.7) and identity $\mathbf{x} \times \mathbf{x} = 0$ can be written in the form

$$\mathbf{M}_k = \frac{1}{2-\theta} (\mathbf{x} \times \mathbf{p}_k). \quad (6.9)$$

Particularly $\mathbf{M}_0 = \mathbf{x} \times \mathbf{p}_0$.

4°. *Center-of-mass theorem.* Center-of-mass theorem of the N -body problem of relativistic mechanics is associated with the Lorentz transformations. If $N = 1$, this theorem is equivalent to the statement that the conservation equation $\frac{d\mathbf{Q}_0}{dt} = 0$ for the vector

$$\mathbf{Q}_0 = \frac{m}{\sqrt{1 - \beta^2}} (\mathbf{x} - t\mathbf{v})$$

is valid. In the de Sitter space the analogous vector-valued C -quantity is given by the formula

$$\mathbf{Q}_k = \frac{m}{\theta \sqrt{1 - \beta^2}} (\mathbf{x} - t\mathbf{v}). \quad (6.10)$$

§ 7 Unsteady transonic gas motion equation

For the unsteady transonic gas motion equation

$$-\Phi_x \Phi_{xx} - 2\Phi_{xt} - \Phi_{yy} = 0 \quad (7.1)$$

the Lagrangian function can be taken in the form

$$\mathcal{L} = -\frac{1}{6} \Phi_x^3 - \Phi_x \Phi_t + \frac{1}{2} \Phi_y^2. \quad (7.2)$$

Equation (7.1) admits the infinite group G_∞ with the tangent vectors

$$(\xi^1, \xi^2, \xi^3, \eta)$$

which depend on five arbitrary functions of time $f(t), g(t), h(t), \sigma(t), \tau(t)$ and have the coordinates

$$\begin{aligned} \xi^1 &= 3f(t), \\ \xi^2 &= f'(t)x + f''(t)y^2 + g'(t)y + h(t), \quad (f' \equiv \frac{df(t)}{dt}, \dots), \\ \xi^3 &= 2f'(t)y + g(t), \\ \eta &= -f'(t)\Phi + f''(t)x^2 + 2f'''(t)xy^2 + \frac{1}{3}f^{(4)}(t)y^4 + 2g''(t)xy \\ &\quad + \frac{2}{3}g'''(t)y^3 + 2h'(t)x + 2h''(t)y^2 + \sigma(t)y + \tau(t). \end{aligned} \quad (7.3)$$

The group G_∞ occurs to be an example of divergence transformations (Remark 13.5). For example, let us consider the subgroup of the group G_∞ generated by the tangent vector

$$\xi^1 = 0, \quad \xi^2 = 0, \quad \xi^3 = 0, \quad \eta = \sigma(t)y. \quad (7.4)$$

The conservation equations will be written in the form

$$D_t C^1 + D_x C^2 + D_y C^3 = 0.$$

In accordance with (3.1)

$$A^1 = -\sigma y \Phi_x, \quad A^2 = -\sigma y \left(\Phi_t + \frac{1}{2} \Phi_x^2 \right), \quad A^3 = \sigma y \Phi_y.$$

The left-hand side of Equation (5.1) for (7.2), (7.4) is equal to

$$-\sigma' y \Phi_x + \sigma \Phi_y = D_x(-\sigma' y \Phi) + D_y(\sigma \Phi),$$

so on the right-hand side of Equation (5.1) one can take the vector B with the components

$$B^1 = 0, \quad B^2 = -\sigma' y \Phi, \quad B^3 = \sigma \Phi.$$

Substitution of these vectors A and B in (5.2) yields the conservation equation

$$D_t(-\sigma y \Phi_x) + D_x \left(-\sigma y \Phi_t - \frac{1}{2} \sigma y \Phi_x^2 + \sigma' y \Phi \right) + D_y(\sigma y \Phi_y + \sigma \Phi) = 0. \quad (7.5)$$

The analogous calculations, carried out for the general tangent vector (7.3) of the group G_∞ , yield the C -vector depending on five arbitrary functions of time. The coordinates of this vector are

$$\begin{aligned} C^1 = & -\frac{1}{2} f \Phi_x^3 + (f'x + f''y^2 + g'y + h) \Phi_x^2 + \frac{3}{2} f \Phi_y^2 + (2f'y + g) \Phi_x \Phi_y \\ & + (f'\Phi - f''x^2 - 2f'''xy^2 - \frac{1}{3} f^{(4)}y^4 - 2g''xy - \frac{2}{3} g'''y^3 - 2h'x \\ & - 2h''y^2 - \sigma y - \tau) \Phi_x + 2(f''x + f'''y^2 + g''y + h') \Phi, \end{aligned}$$

$$\begin{aligned} C^2 = & \frac{1}{3} (f'x + f''y^2 + g'y + h) (\Phi_x^3 + \frac{3}{2} \Phi_y^2) + (3f\Phi_t + 2f'y\Phi_y + g\Phi_y \\ & + f'\Phi - f''x^2 - 2f'''xy^2 - \frac{1}{3} f^{(4)}y^4 - 2g''xy - \frac{2}{3} g'''y^3 - 2h'x \\ & - 2h''y^2 - \sigma y - \tau) (\Phi_t + \frac{1}{2} \Phi_x^2) - \frac{1}{2} f''\Phi^2 + (f'''x^2 + 2f^{(4)}xy^2 \\ & + \frac{1}{3} f^{(5)}y^4 + 2g'''xy + \frac{2}{3} g^{(4)}y^3 + 2h''x + 2h'''y^2 + \sigma'y + \tau') \Phi, \end{aligned}$$

$$\begin{aligned}
C^3 = & -\left(f'y + \frac{1}{2}\right)\left(\frac{1}{3}\Phi_x^3 + \Phi_y^2 + 2\Phi_x\Phi_t\right) - (f'x + f''y^2 + g'y + h)\Phi_x\Phi_y \\
& - 3f\Phi_y\Phi_t + (f''x^2 + 2f'''xy^2 + \frac{1}{3}f^{(4)}y^4 - f'\Phi + 2g''xy \\
& + \frac{2}{3}g'''y^3 + 2h'x + 2h''y^2 + \sigma y + \tau)\Phi_y - 2(2f'''xy + \frac{2}{3}f^{(4)}y^3 \\
& + g''x + g'''y^2 + 2h''y + \frac{1}{2}\sigma)\Phi.
\end{aligned}$$

§ 8 Dirac equations

The Dirac equations

$$\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi = 0 \quad (8.1)$$

together with the conjugate equations

$$\frac{\partial \tilde{\psi}}{\partial x^k} \gamma^k - m\tilde{\psi} = 0 \quad (8.2)$$

are derivable from the variational principle with the Lagrangian function

$$\mathcal{L} = \frac{1}{2} \left\{ \tilde{\psi} \left(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi \right) - \left(\frac{\partial \tilde{\psi}}{\partial x^k} \gamma^k - m\tilde{\psi} \right) \psi \right\}. \quad (8.3)$$

Here ψ is a complex 4-dimensional column vector, γ^k are Dirac's matrices

$$\begin{aligned}
\gamma^1 &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, & \gamma^2 &= \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \\
\gamma^3 &= \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}, & \gamma^4 &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix},
\end{aligned}$$

and $\tilde{\psi}$ is a line-vector defined by the formula

$$\tilde{\psi} = \bar{\psi}^T \gamma^4, \quad (8.4)$$

where $\bar{\psi}^T$ is a line-vector, complex conjugate of the vector ψ .

The Dirac equations give us an example of the degenerate variational problem. To illustrate the difference between Theorem13.1 and Theorem13.2

let us consider the following simple group of transformations admissible by the Dirac equations. Let G_∞ be infinite group with the generator

$$X = \xi^k \frac{\partial}{\partial x^k} + \eta^k \frac{\partial}{\partial \psi^k} + \zeta_k \frac{\partial}{\partial \tilde{\psi}_k} \quad (8.5)$$

with

$$\xi^k = 0, \quad \eta^k = \chi^k(x), \quad \zeta_k = \tilde{\chi}_k(x).$$

Here vectors $\chi(x) = (\chi^1(x), \dots, \chi^4(x))$ and $\tilde{\chi}(x) = (\tilde{\chi}_1(x), \dots, \tilde{\chi}_4(x))$ are connected by means of (8.4) and satisfy the Dirac equations (8.1) and (8.2). The variational integral (2.2) with the Lagrangian function (8.3) is E -invariant with respect to the group G_∞ . It is easy to verify that the variational integral is not invariant with respect to this group. According to Theorem 13.2, substitution of the expressions (8.3), (8.5) into (3.1) gives the C -vector A_∞ with the components

$$A_\infty^k = \tilde{\psi} \gamma^k \chi(x) - \tilde{\chi}(x) \gamma^k \psi, \quad k = 1, \dots, 4.$$

It is well-known, the Dirac equations with the zero rest mass admit the 15-parameter conformal group. This group is generated by the operator (8.5), where $\xi = (\xi^1, \dots, \xi^n)$ is the solution of the generalized Killing equations

$$\frac{\partial \xi^j}{\partial x^k} + \frac{\partial \xi^k}{\partial x^j} = \mu(x) \delta^{jk}, \quad j, k = 1, \dots, 4$$

and

$$\eta = S\psi, \quad \zeta = \tilde{\psi} \tilde{S},$$

where

$$S = \frac{1}{8} \sum_{j,k=1}^4 \frac{\partial \xi^j}{\partial x^k} (\gamma^j \gamma^k - \gamma^k \gamma^j - 3\delta^{jk}), \quad \tilde{S} = \gamma^4 \bar{S}^T \gamma^4.$$

The invariance of the Dirac equations with respect to the conformal group yields the following 15 independent C -vectors A_k, A_{kl} ($k < l$), B_k , $k, l = 1, \dots, 4$, A_0 with the components

$$A_k^j = \frac{1}{2} \left\{ \frac{\partial \tilde{\psi}}{\partial x^k} \gamma^j \psi - \tilde{\psi} \gamma^j \frac{\partial \psi}{\partial x^k} + \delta_k^j \left(\tilde{\psi} \gamma^l \frac{\partial \psi}{\partial x^l} - \frac{\partial \tilde{\psi}}{\partial x^l} \gamma^l \psi \right) \right\}, \quad (8.6)$$

$$A_{kl}^j = \frac{1}{4} \left\{ \tilde{\psi} (\gamma^j \gamma^k \gamma^l + \gamma^k \gamma^l \gamma^j) \psi \right\} + x^l A_k^j - x^k A_l^j, \quad (8.7)$$

$$B_k^j = 2x^l A_{lk}^j + |x|^2 A_k^j, \quad |x|^2 = \sum_{j=1}^4 (x^j)^2, \quad (8.8)$$

$$A_0^j = x^k A_k^j. \quad (8.9)$$

Besides, the Dirac equations with the arbitrary rest mass m admit 4-parameter group, generated by the following one-parameter groups of transformations:

$$\begin{aligned} \psi' &= a\psi, & \psi' &= \psi e^{-ia}, \\ \psi' &= \psi \cosh a + \gamma^4 \gamma^2 \tilde{\psi}^T \sinh a, & \psi' &= \psi \cosh a + i\gamma^4 \gamma^2 \tilde{\psi}^T \sinh a. \end{aligned} \quad (8.10)$$

In this and later formulae only the transformations of the vector ψ are written. The transformations of the vector $\tilde{\psi}$ are defined due to the formula (8.4).

The Dirac equations with the zero rest mass admit the 4-parameter Pauli group as well. This group is generated by the following one-parameter groups of transformations:

$$\begin{aligned} \psi' &= \psi e^{ia\gamma^5}, & \psi' &= \psi e^{-a\gamma^5}, \\ \psi' &= \psi \cos a + \gamma^3 \gamma^1 \tilde{\psi}^T \sin a, & \psi' &= \psi \cos a + i\gamma^3 \gamma^1 \tilde{\psi}^T \sin a, \end{aligned}$$

where $\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$.

The transformations mentioned above together with the group G_∞ occur to be *the main group* of the Dirac equations i.e. the most general continuous local group of transformations admissible by the Dirac equations (8.1), (8.2) [79]. The variational integral is E -invariant with respect to this group, except for the first transformation from (8.10). Not all the C -vectors being obtained with this group, are independent. Namely, the C -vectors associated with three last Pauli transformations appear to be identically zero. Other transformations yield four independent C -vectors C_k , $k = 1, \dots, 4$, with the components

$$C_1^j = -i\tilde{\psi}\gamma^j\psi, \quad C_2^j = \frac{1}{2}(\tilde{\psi}\gamma^j\gamma^4\gamma^2\tilde{\psi}^T - \psi^T\gamma^4\gamma^2\gamma^j\psi),$$

$$C_3^j = \frac{i}{2}(\tilde{\psi}\gamma^j\gamma^4\gamma^2\tilde{\psi}^T + \psi^T\gamma^4\gamma^2\gamma^j\psi), \quad C_4^j = i\tilde{\psi}\gamma^j\gamma^5\psi.$$

If $m \neq 0$, the full list of C -vectors consists of

$$A_\infty, A_k, A_{kl}, \quad k, l = 1, \dots, 4, \quad C_1, C_2, C_3.$$

§ 9 Conservation laws of fluid dynamics

9.1 The ideal gas

Let us consider the equations governing the motion of the ideal gas,

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p &= 0, \\ \rho_t + \mathbf{v} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{v} &= 0, \\ p_t + \mathbf{v} \cdot \nabla p + \gamma p \operatorname{div} \mathbf{v} &= 0, \quad \gamma \equiv \frac{c_p}{c_v} = \text{const.} \end{aligned} \quad (9.1)$$

The main group of Equation (9.1) is known to be wider than the main group of an arbitrary perfect gas motion equations [138]. It is also known that when

$$\gamma = \frac{n+2}{n} \quad (9.2)$$

the further extension of admissible group takes place. Here $n = 1, 2, 3$ respectively for one-dimensional, plane and three-dimensional flows. Now we shall apply the group properties of Equations (9.1) and general conservation theorems in the following way. Firstly let us consider the particular case, namely, the isentropic potential flow of the ideal gas. In this case we can consider the second order equation for the potential $\Phi(t, \mathbf{x})$,

$$\Phi_{tt} + 2\nabla\Phi \cdot \nabla\Phi_t + \nabla\Phi \cdot (\nabla\Phi \cdot \nabla)\nabla\Phi + (\gamma - 1)(\Phi_t + \frac{1}{2}|\nabla\Phi|^2)\Delta\Phi = 0 \quad (9.3)$$

instead of Equations (9.1). Equation (9.3) is derivable by the application of the variational procedure to the variational integral with the Lagrangian function

$$\mathcal{L} = \left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 \right)^{\frac{\gamma}{\gamma-1}}.$$

Having calculated the main group of Equation (9.3) and due to the Noether theorem we shall find conservation laws for Equation (9.3). Then we rewrite the obtained conservation laws in the variables \mathbf{v} , p , ρ using the definition of the potential Φ , the Lagrange-Cauchy integral which can be taken in the form

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} = 0$$

and the relation between pressure p and density ρ in the isentropic flow of the ideal gas. Below the conservation laws are given which have been found by the mentioned above way. Here the integral form of the conservation

equations is used, which is equivalent to the differential form (2.1). The following symbols are used:

- $\Omega(t)$ - arbitrary n -dimensional volume, moving with fluid,
- $S(t)$ - boundary of the volume $\Omega(t)$,
- $\boldsymbol{\nu}$ - unit (outer) normal vector to the surface $S(t)$.

The invariance of Equation (9.3) with respect to the one-parameter group of transformations $\Phi' = \Phi + a$ and the Galileian group (translations in the time, translations of the axes, 3-parameter group of rotations of the axes and three-parameter group of velocity transformations) yield the following classical conservation equations.

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega(t)} \rho d\omega &= 0 && - \text{Conservation of mass} \\
 \frac{d}{dt} \int_{\Omega(t)} \left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{p}{\gamma - 1} \right) d\omega &= - \int_{S(t)} p \mathbf{v} \cdot \boldsymbol{\nu} dS && - \text{Energy} \\
 \frac{d}{dt} \int_{\Omega(t)} \rho \mathbf{v} d\omega &= - \int_{S(t)} \rho \boldsymbol{\nu} dS && - \text{Momentum} \\
 \frac{d}{dt} \int_{\Omega(t)} \rho (\mathbf{x} \times \mathbf{v}) d\omega &= - \int_{S(t)} p (\mathbf{x} \times \boldsymbol{\nu}) dS && - \text{Angular momentum} \\
 \frac{d}{dt} \int_{\Omega(t)} \rho (t\mathbf{v} - \mathbf{x}) d\omega &= - \int_{S(t)} tp \boldsymbol{\nu} dS && - \text{Center-of-mass theorem.}
 \end{aligned}$$

The following additional conservation equations are valid only for the ideal gas, satisfying the condition (9.2),

$$\frac{d}{dt} \int_{\Omega(t)} \{ t(\rho |\mathbf{v}|^2 + np) - \rho \mathbf{x} \cdot \mathbf{v} \} d\omega = - \int_{S(t)} p (2t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS, \quad (9.4)$$

$$\frac{d}{dt} \int_{\Omega(t)} \{ t^2(\rho |\mathbf{v}|^2 + np) - \rho \mathbf{x} \cdot (2t\mathbf{v} - \mathbf{x}) \} d\omega = - \int_{S(t)} 2tp (t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS. \quad (9.5)$$

The conservation equations (9.4), (9.5) arise due to the invariance of Equation (9.3) with respect to two one-parameter groups generated by the tangent vectors of the form $(\tau, \boldsymbol{\xi}, \eta)$ with

$$\tau = 2t, \quad \boldsymbol{\xi} = 0, \quad \eta = 0$$

and

$$\tau = t^2, \quad \boldsymbol{\xi} = t\mathbf{x}, \quad \eta = \frac{1}{2} |\mathbf{x}|^2$$

respectively, if the condition (9.2) is satisfied.

It is easy to verify that the conservation equations (9.4) and (9.5) are valid for arbitrary flow of the ideal gas, satisfying the condition (9.2). Besides, only for the potential flow of the ideal gas with the arbitrary γ the following conservation equation is valid:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega(t)} \left\{ \frac{2\gamma + n(\gamma - 1)}{\gamma + 1} t \left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{p}{\gamma - 1} \right) - \rho \mathbf{x} \cdot \mathbf{v} \right. \\ \left. - \frac{n(\gamma - 1) - 2}{\gamma + 1} \rho \Phi \right\} d\omega = - \int_{S(t)} p \left(\frac{2\gamma + n(\gamma - 1)}{\gamma + 1} t \mathbf{v} - \mathbf{x} \right) \cdot \boldsymbol{\nu} dS. \end{aligned} \quad (9.6)$$

Under the condition (9.2) Equation (9.6) turns into Equation (9.4).

For three-dimensional flow the condition (9.2) ($\gamma = \frac{5}{3}$) is satisfied for the monatomic gas. So, for the monatomic gas the additional conservation equations (9.4) and (9.5) are valid. The analogous situation, i.e. appearance of additional conservation laws, takes place in classical mechanics (motion of a particle in the Newtonian gravitational field) as well as in quantum mechanics (hydrogen atom).

9.2 Perfect incompressible fluid

For the equations of motion of the perfect incompressible fluid,

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0, \quad \operatorname{div} \mathbf{v} = 0, \quad (9.7)$$

some additional conservation laws also take place. One of these additional conservation laws is valid for potential flow of the fluid and has the form

$$\frac{d}{dt} \int_{\Omega(t)} \left(\frac{n+2}{2} t |\mathbf{v}|^2 - \mathbf{x} \cdot \mathbf{v} - n\Phi \right) d\omega = - \int_{S(t)} p [(n+2)t\mathbf{v} - \mathbf{x}] \cdot \boldsymbol{\nu} dS.$$

In formal way this conservation equation may be obtained from Equation (9.6) assuming $\gamma \rightarrow \infty$.

Now, for any vector \mathbf{v} which satisfies the condition $\operatorname{div} \mathbf{v} = 0$, the equation

$$\int_{\Omega} \mathbf{v} d\omega = \int_S (\mathbf{v} \cdot \boldsymbol{\nu}) \mathbf{x} dS$$

is valid. As a result we have the following generalized form of the conservation law of momentum for the incompressible fluid

$$\frac{d}{dt} \int_{\Omega(t)} (\mathbf{f} \cdot \mathbf{v}) d\omega = - \int_{S(t)} \left\{ p \mathbf{f} \cdot \boldsymbol{\nu} - \left(\frac{d\mathbf{f}}{dt} \cdot \mathbf{x} \right) (\mathbf{v} \cdot \boldsymbol{\nu}) \right\} dS, \quad (9.8)$$

where $\mathbf{f}(t) = (f^1(t), \dots, f^n(t))$ is an arbitrary smooth vector function of time. From the group-theoretical point of view the conservation equation (9.8) occurs to be the consequence of the invariance of Equations (9.7) with respect to the infinite group of transformations, generated by the tangent vector $(\tau, \boldsymbol{\xi}, \boldsymbol{\eta}, \mu)$ where

$$\tau = 0, \quad \boldsymbol{\xi} = \mathbf{f}(t), \quad \boldsymbol{\eta} = \frac{d\mathbf{f}(t)}{dt}, \quad \mu = -\frac{d^2\mathbf{f}(t)}{dt^2} \cdot \mathbf{x}.$$

It is known that in the case $n = 2$ Equations (9.1) become the shallow water theory equations if the condition (9.2) is valid. Namely, by setting

$$p = \frac{1}{2}\rho^2, \quad \rho = gh,$$

where g is the acceleration of gravity and h is the depth of the fluid, we obtain the shallow water theory equations:

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} + g\nabla h &= 0, \\ h_t + \mathbf{v} \cdot \nabla h + h \operatorname{div} \mathbf{v} &= 0. \end{aligned}$$

Now Equations (9.4), (9.5) yield the following conservation equations:

$$\frac{d}{dt} \int_{\Omega(t)} h\{t(|\mathbf{v}|^2 + gh) - \mathbf{x} \cdot \mathbf{v}\} d\omega = - \int_{S(t)} \frac{1}{2} gh^2 (2t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS \quad (9.9)$$

and

$$\frac{d}{dt} \int_{\Omega(t)} h\{t^2(|\mathbf{v}|^2 + gh) - \mathbf{x} \cdot (2t\mathbf{v} - \mathbf{x})\} d\omega = - \int_{S(t)} gth^2 (t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS \quad (9.10)$$

for shallow water theory equations which differ from the classical conservation equations.

§ 10 Variational problems of higher order

For variational problems with the Lagrangian functions depending on the higher-order derivatives the main conservation theorems are valid. Only the formula for the C -vector which is obtained in this case has another form.

Let us consider, in particular, the Lagrangian function depending on the second order derivatives,

$$\mathcal{L} = \mathcal{L}(x, u, \partial u, \partial^2 u). \quad (10.1)$$

The Euler equations in this case are of the form

$$\frac{\delta \mathcal{L}}{\delta u^k} \equiv \frac{\partial \mathcal{L}}{\partial u^k} - D_i \frac{\partial \mathcal{L}}{\partial u_i^k} + D_i D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^k} = 0, \quad k = 1, \dots, m. \quad (10.2)$$

The invariance criterion of the variational integral with the Lagrangian function (10.1) will have the form

$$\begin{aligned} \mathcal{L} D_i \xi^i + \xi^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta^k \frac{\partial \mathcal{L}}{\partial u^k} + (D_i \eta^k - u_j^k D_i \xi^j) \frac{\partial \mathcal{L}}{\partial u_i^k} \\ + (D_i D_j \eta^k - u_{jl}^k D_i \xi^l - u_{il}^k D_j \xi^l - u_l^k D_i D_j \xi^l) \frac{\partial \mathcal{L}}{\partial u_{ij}^k} = 0. \end{aligned} \quad (10.3)$$

The left-hand side of Equation (10.3) can be rewritten in the form

$$(\eta^k - u_j^k \xi^j) \frac{\delta \mathcal{L}}{\delta u^k} + D_i A^i, \quad (10.4)$$

where

$$A^i = (\eta^k - u_l^k \xi^l) \left(\frac{\partial \mathcal{L}}{\partial u_i^k} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^k} \right) + D_j (\eta^k - u_l^k \xi^l) \frac{\partial \mathcal{L}}{\partial u_{ij}^k} + \mathcal{L} \xi^i. \quad (10.5)$$

So we have the C -vector (10.5) for every one-parameter group of transformations with the tangent vector (2.8), if the variational integral with the Lagrangian function (10.1) is invariant with respect to this group. It is just the essence of the Noether's theorem for invariant variational problems with the Lagrangian functions of the form (10.1). In this case Theorem 13.2, as well as Theorem 13.3 (by the corresponding change of Definition 13.2), is valid with C -vector (10.5) instead of the vector defined by the formula (3.1).

Let us consider, as an example, the well-known fourth order equation

$$u_{tt} + \Delta^2 u = 0,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The Lagrangian function is

$$\mathcal{L} = \frac{1}{2}(u_t^2 - (\Delta u)^2).$$

Taking the one-parameter group of translations in time we obtain from the formula (10.5) the following conservation equation:

$$\frac{\partial}{\partial t} \{u_t^2 + (\Delta u)^2\} + \operatorname{div} \{2u_t \nabla(\Delta u) - \Delta u \nabla u_t\} = 0.$$

Besides, the invariance of the variational integral with respect to the one-parameter group of rotations with the tangent vector

$$(0, \boldsymbol{\xi}) \quad \text{where} \quad \boldsymbol{\xi} = (y, -x),$$

leads to the conservation equation

$$\frac{\partial}{\partial t}(\theta u_t) + \operatorname{div} \{\theta \nabla(\Delta u) - \Delta u \nabla \theta + \mathcal{L}\boldsymbol{\xi}\} = 0,$$

where $\theta = xu_y - yu_x$. The variational integral is invariant under the translations of the axes as well. Denoting $x = x^1$, $y = x^2$ we can write the conservation equations, associated with the 2-parameter group of translations

$$x'^i = x^i + a^i, \quad i = 1, 2,$$

in the form

$$\frac{\partial \tau^i}{\partial t} + \frac{\partial \eta^{ij}}{\partial x^j} = 0, \quad i = 1, 2,$$

where

$$\tau^i = u_t u_i, \quad \eta^{ij} = u_i \Delta u_j - u_{ij} \Delta u, \quad i, j = 1, 2.$$

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Paper 14

Groups of Lie-Bäcklund contact transformations

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In the theory of contact transformations developed by Sophus Lie [117], [119], [121], one considers a function of several variables and studies continuous groups of transformations preserving first order contact. Generalizations of Lie's theory to the case of several functions or the case of transformations preserving contact of finite order ≥ 2 turns out to be trivial (Theorem 14.1). A nontrivial generalization of Lie's theory is obtained by considering transformations preserving infinite order contact; in this case the number of functions can be arbitrary. The idea of using transformations which preserve infinite order contact of surfaces appears in the work of A. Bäcklund [13]. In the present work we realize this idea in a more general setting, and we examine continuous local Lie groups of transformations preserving infinite order contact, which we call groups of Lie-Bäcklund contact transformations. The infinitesimal characteristics of Lie-Bäcklund groups, formulated in Theorem 14.2, reduce to a result of Lie in the case of groups of first order contact transformations, and to extension formulas in the case of groups of point transformations. We note that the study of Lie-Bäcklund contact transformations gives a group-theoretic basis for a generalization of the methods of group analysis of differential equations [138], as proposed independently by various authors [161], [8].

Let us consider the space of variables $(x, u, u_{(1)}, \dots, u_{(n)})$ where

$$x \in R^N, \quad u \in R^M, \quad u_{(s)} = \{u_{i_1 \dots i_s}^\alpha \mid \alpha = 1, \dots, M; \quad i_1, \dots, i_s = 1, \dots, N\},$$

$s = 1, \dots, n$; the quantities $u_{i_1 \dots i_s}^\alpha$ are symmetric in all lower indices. Let G be a group of transformations

$$\begin{aligned} x'^i &= f^i(x, u, u_{(1)}, \dots, u_{(n)}; a) \\ u'^\alpha &= \varphi^\alpha(x, u, u_{(1)}, \dots, u_{(n)}; a) \\ u_i'^\alpha &= \psi_i^\alpha(x, u, u_{(1)}, \dots, u_{(n)}; a) \\ &\dots\dots\dots \\ u_{i_1 \dots i_n}^\alpha &= \psi_{i_1 \dots i_n}^\alpha(x, u, u_{(1)}, \dots, u_{(n)}; a), \end{aligned} \quad (1)$$

$\alpha = 1, \dots, M$; $i, i_1, \dots, i_n = 1, \dots, N$, in this space. Without loss of generality we may assume that G is a one-parameter group.

Definition 14.1. The group G is a *group of n th-order contact transformations* if the equations

$$du^\alpha - u_j^\alpha dx^j = 0, \quad du_{i_1 \dots i_s}^\alpha - u_{i_1 \dots i_s j}^\alpha dx^j = 0, \quad (2)$$

$$s = 1, \dots, n-1; \quad \alpha = 1, \dots, M; \quad i_1, \dots, i_s = 1, \dots, N,$$

give an invariant manifold of the group \tilde{G} obtained by extending the transformations (1) to the differentials $dx^i, du^\alpha, \dots, du_{i_1 \dots i_n}^\alpha$.

One example of a group of n th-order contact transformations is a group of point transformations

$$x'^i = g^i(x, u; a), \quad i = 1, \dots, N; \quad u'^\alpha = h^\alpha(x, u; a), \quad \alpha = 1, \dots, M, \quad (3)$$

which is extended by the usual change of variables formulas to derivatives of u with respect to x up to order n . Another example is the Lie contact transformations which correspond to the case $M = 1$ and are the contact transformations of the first order. An extension of a Lie group of contact transformations to derivatives up to order n provides a group of n th-order contact transformations. The following theorem shows that these examples exhaust the whole class of contact transformations of order n for any n .

Theorem 14.1. Every group of n th order contact transformations is either

(I) a group of pointwise transformations (3) extended to derivatives up to order n if $M > 1$, or

(II) a Lie group of contact transformations extended to derivatives up to order n if $M = 1$.

In [13], while examining the case of a function u of several variables x^1, \dots, x^N , A. Bäcklund studied transformations of the variables x, u depending not only on these variables, but also on derivatives of u with respect to x up to some finite order. He interpreted these transformations as transformations of surfaces in (x, u) -space and introduced the notion of transformations preserving infinite order contact of these surfaces. In addition he found that transformations of the derivatives of any finite order necessarily depend on higher order derivatives. Taking this fact and Theorem 14.1 into account, we will examine groups of transformations in the infinite-dimensional space of variables $(x, u, u_{(1)}, u_{(2)}, \dots)$ under the assumption that these transformations preserve infinite order contact.

Let $x \in R^N$, $u \in R^M$, and for each $s = 1, 2, \dots$ let $u_{(s)}$ be defined as above. We examine the analytic scale \mathcal{A} [141], [142] of elements $z = (x, u, u_{(1)}, u_{(2)}, \dots)$ and a one-parameter local Lie group G of transformations

$$z' = F(z; a) \quad (4)$$

in \mathcal{A} , which will be written in coordinate form as follows:

$$\begin{aligned} x'^i &= f^i(x, u, u_{(1)}, \dots; a), \\ u'^\alpha &= \varphi^\alpha(x, u, u_{(1)}, \dots; a), \\ u'_i{}^\alpha &= \psi_i^\alpha(x, u, u_{(1)}, \dots, a), \\ &\dots\dots\dots \end{aligned}$$

Let \tilde{G} be the extension of the group G to differentials $dz = (dx^i, du^\alpha, du_i^\alpha, \dots)$ given by the formulas

$$\begin{aligned} dx'^i &= \frac{\partial f^i}{\partial x^j} dx^j + \frac{\partial f^i}{\partial u^\beta} du^\beta + \frac{\partial f^i}{\partial u_j^\beta} du_j^\beta + \dots, \\ du'^\alpha &= \frac{\partial \varphi^\alpha}{\partial x^j} dx^j + \frac{\partial \varphi^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \varphi^\alpha}{\partial u_j^\beta} du_j^\beta + \dots, \\ du'_i{}^\alpha &= \frac{\partial \psi_i^\alpha}{\partial x^j} dx^j + \frac{\partial \psi_i^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \psi_i^\alpha}{\partial u_j^\beta} du_j^\beta + \dots, \\ &\dots\dots\dots \end{aligned}$$

Definition 14.2. The group G is a group of Lie-Bäcklund contact transformations if the system of equations

$$du^\alpha - u_j^\alpha dx^j = 0, \quad du_{i_1 \dots i_s}^\alpha - u_{i_1 \dots i_s j}^\alpha dx^j = 0, \quad s = 1, 2, \dots, \quad (5)$$

$$\alpha = 1, \dots, M; \quad i_1, \dots, i_s = 1, \dots, N,$$

is invariant relative to the extended group \tilde{G} .

Now suppose that we are given a one-parameter family of transformations (4). We define the mapping $\Xi : \mathcal{A} \rightarrow \mathcal{A}$ by the formula

$$\Xi(z) = \left. \frac{\partial F(z; a)}{\partial a} \right|_{a=0} \quad (6)$$

and we assume that the ordinary differential equation with initial condition

$$\frac{dz}{da} = \Xi(z), \quad z|_{a=0} = z_0$$

has a unique solution $z \in \mathcal{A}$; one can assume, e.g. that Ξ is quasi-differential [141], [142]. Under this assumption the group of transformations (4) is uniquely determined by its vector field (6). That is, the mapping $F(z, a)$ gives a one-parameter local Lie group if and only if it is a solution of the Lie equations

$$\frac{dF}{da} = \Xi(F), \quad F|_{a=0} = z. \quad (7)$$

The following theorem gives an infinitesimal characterization of groups of Lie-Bäcklund contact transformations; it makes use of the standard notation

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ii_1}^\alpha \frac{\partial}{\partial u_{i_1}^\alpha} + u_{ii_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots \quad (8)$$

and coordinate notation

$$\Xi = (\xi^i, \eta^\alpha, \zeta_i^\alpha, \zeta_{i_1 i_2}^\alpha, \dots) \quad (9)$$

for the vector field (6).

Theorem 14.2. The group of transformations (4) is a group of Lie-Bäcklund transformations if and only if the vector field (9) satisfies the conditions

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\alpha) - u_{i_1 \dots i_{s-1} j}^\alpha D_{i_s}(\xi^j), \quad s = 2, 3, \dots, \\ \alpha &= 1, \dots, M; \quad i, i_1, \dots, i_s = 1, \dots, N. \end{aligned} \quad (10)$$

Using the relations (10), one may establish that groups of Lie-Bäcklund contact transformations which are neither groups of point transformations (3) nor Lie groups of contact transformations do not have finite-dimensional invariant subspaces in \mathcal{A} ; in fact the Lie equations (7) represent an essentially infinite system of equations. Specifically, we have the following result.

Theorem 14.3. Suppose a group of Lie-Bäcklund contact transformations (4) is an extension of the group G of transformations (1) for some $n \geq 1$. Then G is the n -th extension either of a group of point transformations or of a Lie group of contact transformations.

Lie-Bäcklund contact transformations may be used for a group-theoretic interpretation of a generalization [161], [8] of group methods of analysis of differential equations. The point of this generalization is that, instead of considering infinitesimal operators of groups of point transformations that are first order linear differential operators, one introduces higher order linear differential operators. This generalization leaves unanswered the question of assigning to these operators a natural transformation group. We now see that operators of higher order (both linear and nonlinear) can be obtained by considering infinitesimal operators of groups of Lie-Bäcklund contact transformations. In fact, let the functions $u'^\alpha(x)$ be obtained from the functions $u^\alpha(x)$ as a result of the transformations (4). Standard calculations give

$$u'^\alpha(x) = u^\alpha(x) + aQ^\alpha[u(x)] + o^\alpha(a),$$

where Q^α is the differential operator

$$Q^\alpha[u(x)] = \eta^\alpha \left(x, u(x), \frac{\partial u^\beta(x)}{\partial x^j}, \dots \right) - \xi^i \left(x, u(x), \frac{\partial u^\beta(x)}{\partial x^j}, \dots \right) \frac{\partial u^\alpha(x)}{\partial x^i}.$$

If, in particular,

$$\eta^\alpha - \xi^i u_i^\alpha = q^\alpha(x) + q_\beta^\alpha(x) u^\beta + \sum_{s=1}^{\infty} q_\beta^{\alpha, j_1 \dots j_s}(x) u_{j_1 \dots j_s}^\beta,$$

then Q^α will be the linear differential operator

$$Q^\alpha[u(x)] = q^\alpha(x) + q_\beta^\alpha(x) u^\beta(x) + \sum_{s=1}^{\infty} q_\beta^{\alpha, j_1 \dots j_s}(x) \frac{\partial^s u^\beta(x)}{\partial x^{j_1} \dots \partial x^{j_s}},$$

considered in the generalization indicated above.

The concept of a group of Lie-Bäcklund contact transformations admitted by a given system of differential equations and the method for determining such a group are formulated the same as in the case of a group of point transformations, with the difference that when considering the Lie-Bäcklund group it is necessary to use the original differential equations together with their differential consequences. By Theorem 14.2, the contact vector field (9) of the Lie-Bäcklund group is determined by its coordinates ξ^i , $i = 1, \dots, N$, and η^α , $\alpha = 1, \dots, M$. If we write the invariance criterion for

the given system of differential equations relative to the Lie-Bäcklund group in terms of the vector field (9), then taking account of formula (10) we obtain equations in the functions $\xi^i(x, u, u_{(1)}, \dots)$ and $\eta^\alpha(x, u, u_{(1)}, \dots)$ known as the *determining equations* for the desired Lie-Bäcklund group.

As an example we consider the Schrödinger equation for the hydrogen atom

$$\frac{1}{2} \Delta u + \left(\frac{1}{r} - k \right) u = 0,$$

where $r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}$ and k is a constant. It admits a 3-parameter Lie-Bäcklund group generated by the tangent vectors $\Xi_i = (\xi_{(i)}^j, \eta_{(i)}, \dots)$, $i = 1, 2, 3$, with coordinates

$$\xi_{(i)}^j = \delta_i^j, \quad \eta_{(i)} = - \sum_{j,l,m,n=1}^3 e_{ijl} e_{jmn} x^m u_{nl} + \frac{x^i}{r} u,$$

where δ_i^j is the Kronecker symbol and e_{ijl} is the usual permutation symbol. The existence of this group leads to the specific conservation law for the hydrogen atom known as the Lenz-Runge vector. This group is clearly not a Lie group of contact transformations. We note that the equation under consideration does not in general admit Lie contact transformations different from point transformations.

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Paper 15

Lie-Bäcklund groups and conservation laws

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The existence of a connection between the conservation laws for differential equations obtained from a variational principle and the invariance of the corresponding variational problem was established in the works of Jacobi, Klein, and Noether. Jacobi [99] described such a connection for the equations of classical mechanics. Klein [101], [102] established a similar fact for the equations of the general theory of relativity and proposed a study of the group-theoretic basis of the conservation laws for arbitrary differential equations. In [133], Noether, under the direct influence of Klein's ideas, obtained a sufficient condition for the existence of conservation laws for arbitrary differential equations obtained from a variational principle, based on the invariance properties of the corresponding variational problem. This result, now known as Noether's theorem, says that if the values of a variational integral for arbitrary admissible functions are invariant with respect to an r -parameter continuous group of transformations of the dependent and independent variables, then the Euler equations for the extremals of the functional under consideration have r linearly independent conservation laws. All of these conservation laws can be obtained by a certain standard formula, and the order of the derivatives on which the conservation laws depend does not exceed the largest order of the derivatives which appear in the corresponding Lagrange function. The restrictions mentioned are essential, and while Noether's theorem is sufficiently general to deal with the

special equations of mechanics and physics, it leaves open the question of the group-theoretic nature of conservation laws for arbitrary systems of differential equations. Furthermore, the invariance condition on the functionals in variational problems, which is an assumption in Noether's theorem, is not necessary for the existence of conservation laws. A discussion of this question and extensive bibliographies can be found in [75], [79], [81].

In the present paper, Klein's idea that conservation laws have a group-theoretic character is established for arbitrary systems of differential equations. The concept of invariance under Lie-Bäcklund groups of contact transformations (see [90]) plays a fundamental role.

First of all, we treat the question of determining groups of Lie-Bäcklund tangent transformations admissible by the differential equations, in order to avoid certain formal difficulties in the calculation of these groups. To simplify the notation, we consider only one-parameter groups.

Let G be a locally continuous group of transformations

$$z' = f(z, a) \quad (1)$$

in a space S whose elements are sequences $z = (x, u, u_{(1)}, u_{(2)}, \dots)$, where $x = (x^1, \dots, x^n) \in R^n$, $u = (u^1, \dots, u^m) \in R^m$, and for each natural number ν let $u_{(\nu)}$ denote a set of quantities $u_{i_1 \dots i_\nu}^\alpha$ ($\alpha = 1, \dots, m$; $i_1, \dots, i_\nu = 1, \dots, n$) which are symmetric in all of the lower indices. We assume that there is a norm defined in S and that it is a Banach space under the natural structure of a real vector space. A group G is called a *Lie-Bäcklund group of contact transformations*, or, briefly, a *Lie-Bäcklund group* if its extension to differentials dz preserves "contact of infinite order" [90]. If the tangent vector field $\Xi : S \rightarrow S$ of the group G , defined by

$$\left. \frac{\partial f(z, a)}{\partial a} \right|_{a=0} = \Xi(z) = (\xi^i(z), \eta^\alpha(z), \zeta_i^\alpha(z), \zeta_{i_1 i_2}^\alpha(z), \dots),$$

is a smooth mapping, then the transformations (1) of this group are uniquely determined as solutions of the Lie equations $\frac{df}{da} = \Xi(f)$, $f|_{a=0} = z$. The group G is a Lie-Bäcklund group if and only if

$$\zeta_{i_1 \dots i_\nu}^\alpha = D_{i_1} \dots D_{i_\nu}(\mu^\alpha) + \xi^j u_{j i_1 \dots i_\nu}^\alpha, \quad \nu = 1, 2, \dots, \quad (2)$$

where

$$\mu^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\nu=1}^{\infty} u_{i i_1 \dots i_\nu}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_\nu}^\alpha}.$$

Here and in what follows we use the usual rule of summation over repeated indices. The relations (2), which differ in form from the criterion for invariance of contact of infinite order in [90], are convenient to apply.

We consider a system of differential equations of order q :

$$\omega(x, u, u_{(1)}, \dots, u_{(q)}) = 0, \quad (3)$$

where $\omega = (\omega_1, \dots, \omega_N)$. For each $\nu = 1, 2, \dots$, the collection of all expressions $D_{i_1} \dots D_{i_\nu}(\omega_j)$ ($j = 1 \dots, N$; $i_1, \dots, i_\nu = 1, \dots, n$) will be denoted by $\omega_{(\nu)}$. The equations (3) together with all of their differential consequences define a manifold $\Omega \subset S$; this manifold is given by the equations

$$\Omega : \omega(z_q) = 0, \quad \omega_{(1)}(z_{q+1}) = 0, \quad \omega_{(2)}(z_{q+2}) = 0, \dots, \quad (4)$$

where we have introduced the notation $z_\nu = (x, u, u_{(1)}, \dots, u_{(\nu)})$. The system (3) is called *invariant* with respect to a Lie-Bäcklund group G if the manifold Ω is invariant under the transformations (1) of that group [90]; in this case, we will also call G *admissible by the equations* (3). The criterion for invariance of the system (3) with respect to a Lie-Bäcklund group can be written in terms of the infinitesimal operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\nu=1}^{\infty} \zeta_{i_1 \dots i_\nu}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_\nu}^\alpha} \quad (5)$$

of the group in the form of the following infinite sequence of equations:

$$X(\omega)|_\Omega = 0, \quad X(\omega_{(\nu)})|_\Omega = 0, \quad \nu = 1, 2, \dots \quad (6)$$

The coordinates $\zeta_{i_1 \dots i_\nu}^\alpha$, $\nu = 1, 2, \dots$, in (5) can be expressed in terms of the coordinates ξ^i , η^α by means of (2); therefore the conditions (6) are a system of equations in $\xi^i(z)$, $\eta^\alpha(z)$, called *determining equations* for the Lie-Bäcklund group admissible by the equations (3).

Definite difficulties arise in the determination of Lie-Bäcklund groups admissible by concrete differential equations by means of a solution to the equations (6). These difficulties are related to the fact that, in contrast with the case of groups of pointwise transformations [138], the determining equations (6) are not differential equations but are relations between the values of $\xi^i(z)$, $\eta^\alpha(z)$ and their derivatives at points $z \in \Omega$. In addition, the conditions (6) consist of an infinite sequence of relations, which can also lead to additional difficulties. The following considerations allow us to circumvent these difficulties.

We note first of all that (6) is satisfied for any operator X_ω of the form (5) whose coordinates ξ^i , η^α , are linear forms in ω , $\omega_{(1)}, \dots$ with arbitrary coefficients which are regular at the points $z \in \Omega$. Therefore the operator X satisfies the determining equations (6) if and only if the operators $X - X_\omega$ satisfy them. This remark allows us to limit our considerations only to

operators (5) whose coordinates do not depend on (4), in which case (6) will be a system of differential equations.

The following theorem shows that the infinite sequence (6) is a consequence of the equations

$$X(\omega)|_{\Omega} = 0. \quad (7)$$

Theorem 15.1. For any operator (5) satisfying (2), we have the identities

$$D_i X - X D_i = D_i(\xi^j) D_j, \quad i = 1, \dots, n. \quad (8)$$

In what follows, (7) will be called determining equations of the Lie-Bäcklund group admissible by the system of differential equations (3). We note that (7) are satisfied by the operator $X_0 = \xi^i(z) D_i$ with arbitrary $\xi^i(z)$, for any system (3). Therefore when we determine the Lie-Bäcklund group by means of a solution of the determining equations we may assume that the coordinates ξ^i , $i = 1, \dots, n$, of the required operator (5) are zero.

Noether's theorem [133] gives the existence of conservation laws for systems of differential equations (3) if they are Euler equations of a functional

$$l[u] = \int \mathcal{L}(x, u, u_{(1)}, \dots, u_{(q)}) dx$$

which is invariant with respect to some Lie group of pointwise transformations in the space of the variables x, u .

Conserved quantities are obtained from the infinitesimal criterion for invariance of the functional. However, the existence of conservation laws can be established under less restrictive assumptions on the equations (3) (see [79], [81]). The following definition contains all of these assumptions.

Definition 15.1. Suppose that the system (3) of differential equations is admissible by the Lie-Bäcklund group G with infinitesimal operator (5). A smooth mapping $\mathcal{L} : S \rightarrow S$, satisfying the condition

$$\{X(\mathcal{L}) + \mathcal{L} D_i(\xi^i)\}|_{\Omega} = 0, \quad \frac{\delta \mathcal{L}}{\delta u^{\alpha}}|_{\Omega} = 0, \quad \alpha = 1, \dots, m,$$

where

$$\frac{\delta \mathcal{L}}{\delta u^{\alpha}} = \frac{\partial \mathcal{L}}{\partial u^{\alpha}} + \sum_{\nu=1}^{\infty} (-1)^{\nu} D_{i_1 \dots} D_{i_{\nu}} \frac{\partial \mathcal{L}}{\partial u_{i_1 \dots i_{\nu}}^{\alpha}} \quad (9)$$

is called a *relatively G -invariant weak Lagrangian* of the system (3).

The main result concerning the connection between conservation laws and invariance properties of differential equations with respect to Lie-Bäcklund groups can now be stated in the following form.

Theorem 15.2. Suppose that an arbitrary system (3) of differential equations with smooth coefficients is given. There exists a conservation law $D_i(A^i)|_\Omega = 0$ for this system if and only if it admits a Lie-Bäcklund group G and has a relatively G -invariant weak Lagrangian \mathcal{L} . In addition, a vector $A(z) = (A^1(z), \dots, A^n(z))$ satisfying the conservation law is expressible in terms of \mathcal{L} and the infinitesimal operator (5) of G by

$$A^i = \mathcal{L}\xi^i + (\eta^\alpha - \xi^l u_l^\alpha) \left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha} + \sum_{\nu=1}^{\infty} (-1)^\nu D_{j_1} \dots D_{j_\nu} \frac{\partial \mathcal{L}}{\partial u_{ij_1 \dots j_\nu}^\alpha} \right) + \quad (10)$$

$$+ \sum_{s=1}^{\infty} D_{k_1} \dots D_{k_s} (\eta^\alpha - \xi^l u_l^\alpha) \left(\frac{\partial \mathcal{L}}{\partial u_{ik_1 \dots k_s}^\alpha} + \sum_{\nu=1}^{\infty} (-1)^\nu D_{j_1} \dots D_{j_\nu} \frac{\partial \mathcal{L}}{\partial u_{ij_1 \dots j_\nu k_1 \dots k_s}^\alpha} \right).$$

The proof of this theorem is based on the following identity, which is valid for an arbitrary Lie-Bäcklund group with infinitesimal operator (5) and an arbitrary smooth function $\mathcal{L}(x, u, u_{(1)} \dots)$:

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = (\eta^\alpha - \xi^l u_l^\alpha) \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i(A^i),$$

where the expressions $\frac{\delta \mathcal{L}}{\delta u^\alpha}$ and A^i are obtained from (9) and (10). The details are not given here.

The idea behind the relative G -invariance of the weak Lagrangian \mathcal{L} is that the values of the functional $l[u] = \int \mathcal{L}(x, u, u_{(1)}, \dots) dx$, corresponding to it, calculated for any solution $u^a = \phi^a(x)$ of (3), are invariant under the transformations of the group G . Therefore, according to Theorem 15.2, the existence of any conservation law for the equations (3) is the result of the invariance of these values of the functional $l[u]$ with respect to some Lie-Bäcklund group which is admissible by the equations (3).

The operations of addition, multiplication by a number, and differentiation are defined on the set of all vectors which satisfy a conservation law for a given system of differential equations. In addition, for each system of equations it is possible to define other operations which act in this set of vectors by making use of Lie-Bäcklund groups which are admissible by these equations. One such operation is given by (8) and has the form $A \mapsto X(A)$, where X is the operator (5) with $\xi^i = \text{const.}$, $i = 1, \dots, n$, and satisfying (7).

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Paper 16

On the Cauchy problem for the equation $u_{tt} - u_{xx}$

$$- \sum_{i,j=1}^{n-1} a_{ij}(x-t)u_{y_i y_j} = 0$$

N. H. IBRAGIMOV AND E. V. MAMONTOV [93]

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At the present time the theory of linear second-order differential equations of hyperbolic type, in that part of it which is devoted to questions on the existence and uniqueness of the solution of the Cauchy problem, can be considered completed. After Hadamard's work it became clear, however, that besides existence and uniqueness theorems we can also study the "explicit" representations of the solution. Hadamard was the first to give such representations and to demonstrate their importance in a number of questions connected with the behavior of the solutions.

One such question is the problem of seeking the class of all second-order equations for which wave diffusion is absent or, what is the same, for which the Huygens principle is valid. The Huygens principle is valid for a wave equation with an odd number of space variables, not less than three; this follows directly from the Poisson formula in the case of three space variables, and from the representation given by Tedone [170] of the solution of the Cauchy problem in the general case. This circumstance

served Hadamard as a starting point for formulating the Huygens principle for linear second-order hyperbolic equations of general form. Hadamard [61] obtained a criterion for the validity of the Huygens principle for such equations, and proved that wave diffusion always holds in the case of an even number of space variables.

Hadamard's criterion was formulated in terms of the elementary solution of the equation, and can be used only if we succeed in obtaining a more or less explicit, representation of the solution. This explains why the wave equation remained for along time the only known example of an equation without wave diffusion. Furthermore, in the case of three space variables, as Mathisson [128] proved, any equation without wave diffusion with constant coefficients of the second derivatives is equivalent to the wave equation. This is not true in the case of a larger number of variables, as Stellmacher's example shows [164]. For any odd number of space variables, greater than or equal to five, he constructed equations with the wave operator in the principal part, not equivalent to the wave equation and for which the Huygens principle is valid. The equations considered by Stellmacher are the first examples of equations without wave diffusion other than the wave equation. The restrictions indicated on the number of variables are essential in the construction of these examples, and, as Mathisson's theorem shows, in the case of three space variables there cannot be similar examples, i.e. equations without wave diffusion where a principal part is that of the wave equation but not equivalent to the wave equation.

Thus, equations with three space variables, not equivalent to the wave equation, for which wave diffusion is absent must be equations with variable coefficients of the second derivatives. Such examples have been given by Günther [60] and by us [92]. Günther established the validity of the Huygens principle in the examples considered, using Hadamard's criterion, which is formulated particularly simply for the case of three space variables. In [92] this fact followed from an explicit formula for the solution. The examples constructed in the three-dimensional case are based on a geometric foundation: the corresponding equations are unique equations, invariant relative to the group of conformal transformations in four-dimensional Riemannian spaces of hyperbolic type, connected with these equations, where the spaces indicated have a "nontrivial conformal group". Furthermore, an assertion analogous to Mathisson's theorem is valid in the spaces being considered; it can be stated in the following invariant form. *For a second-order equation to which corresponds a Riemannian space with a nontrivial conformal group, the Huygens principle is valid if and only if this equation is invariant relative to the group of conformal transformations; such an equation exists, and, moreover, is unique, in all the spaces being examined.* A discussion of

the corresponding questions and literature references are contained in [79].

It is interesting that in all known examples of equations without wave diffusion explicit formulas have been successfully obtained for the solution of the Cauchy problem, from which follows the validity of the Huygens principle. In the present paper we derive a formula yielding the solution of the Cauchy problem for the given in the title hyperbolic equation with an arbitrary number n of space variables. The validity of the Huygens principle for this equation for any odd $n \geq 3$ follows directly from the formula obtained. When $n = 3$ we obtain the known examples in [60] and [92], while for greater odd values of n we obtain examples of equations without wave diffusion equivalent neither to the wave equation nor to the examples of Stellmacher.

The equation being examined is solved with the aid of the Fourier transform with respect to the variable $y = (y_1, \dots, y_{n-1})$. The Fourier transform of the solution satisfies a differential equation with two independent variables x and t , the solution of whose Cauchy problem can be constructed in explicit form. Next, for the case of odd $n \geq 3$ we put into consideration a generalized function with compact support, and with its help we realize an approach to solving the original problem. In the case of even values $n \geq 2$ the solution is constructed by a descent method.

§ 1 Statement of the problem. Preliminary results

We consider a second-order differential equation of the form

$$L[u] \equiv u_{tt} - u_{xx} - \sum_{i,j=1}^{n-1} a_{ij}(x-t)u_{y_i y_j} = 0 \quad (1.1)$$

with an arbitrary number $n \geq 2$ of space variables (x, y) , where $y = (y_1, \dots, y_{n-1})$ and x is a selected variable. The coefficients a_{ij} of this equation are taken to be arbitrary, symmetric ($a_{ij} = a_{ji}$) and infinitely differentiable functions of only one variable $x - t$, such that equation (1.1) is strictly hyperbolic, i.e. the quadratic form $\sum_{i,j=1}^{n-1} a_{ij}\xi_i\xi_j$ is positive definite.

For the equation mentioned we seek, in the domain $t > 0$, the solution of the Cauchy problem with data at $t = 0$:

$$L[u] = 0, \quad u|_{t=0} = f(x, y), \quad u_t|_{t=0} = g(x, y). \quad (1.2)$$

The initial data are assumed to be infinitely differentiable functions of compact support. These requirements on the equation's coefficients and on

the initial data are adopted only for the convenience of the presentation to follow, and are not necessary for the validity of the final results.

Under the assumptions made, the Cauchy problem (1.2) has an infinitely differentiable solution which is unique. It has compact support with respect to the space variables. We obtain a formula giving a representation of this solution, from which the validity of the Huygens principle for equation (1.1) with an arbitrary odd $n \geq 3$ follows directly.

We can limit the analysis to the following special Cauchy problem:

$$L[u] = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = h(x, y). \quad (1.3)$$

As a matter of fact, the following statement is valid.

Lemma 16.1. Let $v(t, x, y)$ be the solution of the special Cauchy problem (1.3) with $h = f$, and let $w(t, x, y)$ be the solution of (1.3) with $h = f_x - g$. Then the function

$$u = v_t + v_x - w \quad (1.4)$$

is a solution of the Cauchy problem (1.2).

Indeed, the function u defined by (1.4) satisfies equation (1.1), since the operator L commutes with the operator $\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, i.e.

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) L = L \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right).$$

The fulfillment of the initial conditions of problem (1.2) for the function u is verified directly.

Let us apply the Fourier transform with respect to y to obtain the formula giving a representation of the solution of problem (1.3). If $u(t, x, y)$ is the solution of this problem, then its Fourier transform

$$\hat{u}(t, x; \lambda) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{R^{n-1}} u(t, x, y) e^{-i(y, \lambda)} dy \quad (1.5)$$

satisfies the equation

$$\hat{u}_{tt} - \hat{u}_{xx} + \left(\sum_{i,j=1}^{n-1} a_{ij}(x-t) \lambda_i \lambda_j \right) \hat{u} = 0 \quad (1.6)$$

with two independent variables t and x (with parameters $\lambda_1, \dots, \lambda_{n-1}$) and initial data

$$\hat{u}|_{t=0} = 0, \quad \hat{u}_t|_{t=0} = \hat{h}(x; \lambda), \quad (1.7)$$

where \hat{h} is the Fourier transform of $h(x, y)$:

$$\hat{h}(x; \lambda) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{R^{n-1}} h(x, y) e^{-i(y, \lambda)} dy;$$

here $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ and $(y, \lambda) = \sum_{i=1}^{n-1} y_i \lambda_i$.

Let us now write out in explicit form the solution of problem (1.6), (1.7) for any real values of the parameters $\lambda_1, \dots, \lambda_{n-1}$. This proves to be possible because of

Lemma 16.2. The Riemann function of equation (1.6) has the form

$$R(\tau, \xi; t, x) = J_0 \left(\sqrt{(t - \tau + x - \xi) \sum_{i,j=1}^{n-1} [A_{ij}(\xi - \tau) - A_{ij}(x - t)] \lambda_i \lambda_j} \right),$$

where J_0 is a Bessel function and $A_{ij}(\sigma) = \int a_{ij}(\sigma) d\sigma$, $i, j = 1, \dots, n-1$.

Proof. The lemma is trivial if the vector λ equals zero. If $\lambda \neq 0$, then the nonsingular change of variables

$$\bar{t} = \frac{1}{2}(t + x), \quad \bar{x} = -\frac{1}{2} \sum_{i,j=1}^{n-1} A_{ij}(x - t) \lambda_i \lambda_j$$

leads to the equality

$$\hat{u}_{tt} - \hat{u}_{xx} = \left(\sum_{i,j=1}^{n-1} a_{ij}(x - t) \lambda_i \lambda_j \right) \hat{u}_{\bar{t}\bar{x}}.$$

Therefore, in the new variables \bar{t} and \bar{x} equation (1.6) has the form

$$\hat{u}_{\bar{t}\bar{x}} + \hat{u} = 0.$$

The Riemann function for this equation is known; it is equal to

$$J_0 \left(\sqrt{4(\bar{t} - \bar{\tau})(\bar{x} - \bar{\xi})} \right)$$

with $\bar{\tau} = \frac{1}{2}(\tau + \xi)$ and $\bar{\xi} = -\frac{1}{2} \sum_{i,j=1}^{n-1} A_{ij}(\xi - \tau) \lambda_i \lambda_j$. Reverting now to the old variables t, x, τ, ξ , we obtain the lemma's assertion.

To obtain the solution of problem (1.6), (1.7) we substitute the above value of the Riemann function into the general formula

$$\hat{u}(t, x; \lambda) = \frac{1}{2} \int_{x-t}^{x+t} R(0, \xi; t, x) \hat{h}(\xi, \lambda) d\lambda$$

for the solution of this Cauchy problem. Invoking the definition of \hat{h} , we rewrite the formula obtained for the Fourier transform of the solution of Cauchy problem (1.3) as follows:

$$\hat{u}(t, x; \lambda) = \frac{1}{2(2\pi)^{\frac{n-1}{2}}} \int_{x-t}^{x+t} R(0, \xi; t, x) \int_{R^{n-1}} h(\xi, \eta) e^{-i(\lambda, \eta)} d\eta. \quad (1.8)$$

We introduce the notation $k^2 = t + x - \xi$ and $Q(\lambda) = \sum_{i,j=1}^{n-1} [A_{ij}(\xi) - A_{ij}(x-t)] \lambda_i \lambda_j$. In this notation (1.8) takes the form

$$\hat{u}(t, x; \lambda) = \frac{1}{2(2\pi)^{\frac{n-1}{2}}} \int_{x-t}^{x+t} J_0(k\sqrt{Q(\lambda)}) d\xi \int_{R^{n-1}} h(\xi, \eta) e^{-i(\lambda, \eta)} d\eta.$$

The function $u(t, x, y)$ is obtained from $\hat{u}(t, x, y)$ by taking the inverse Fourier transform:

$$u(t, x, y) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{R^{n-1}} \hat{u}(t, x; \lambda) e^{i(y, \lambda)} d\lambda.$$

Let us implement the inverse Fourier transform, having changed the order of integration with respect to λ and ξ . Then

$$u(t, x, y) = \frac{1}{2(2\pi)^{n-1}} \int_{x-t}^{x+t} d\xi \int_{R^{n-1}} J_0(k\sqrt{Q(\lambda)}) d\lambda \int_{R^{n-1}} h(\xi, \eta) e^{-i(\lambda, \eta-y)} d\eta. \quad (1.9)$$

Formula (1.9), yielding a representation of the solution of Cauchy problem (1.3), admits a further simplification. We remark first that the quadratic form $Q(\lambda)$ is positive definite for $\xi > x - t$; indeed, we can write the formula for $Q(\lambda)$ as an integral $Q(\lambda) = \int_{x-t}^{\xi} q(\alpha; \lambda) d\alpha$ of the quadratic form $q(\alpha; \lambda) = \sum_{i,j=1}^{n-1} a_{ij}(\alpha) \lambda_i \lambda_j$ which is positive definite for each value of α . Therefore, there exists the nonsingular real linear transformation

$$\mu = T\lambda, \quad (1.10)$$

taking the form $Q(\lambda)$ into a sum of squares: $Q = \sum_{i=1}^{n-1} \mu_i^2$. In (1.9) we make a change of variables λ in accordance with (1.10). Here it turns out to be convenient to simultaneously transform the integration variables η . Let T' denote the transposed matrix of transformation (1.10). Instead of the variables η we introduce the new integration variables $z = T'^{-1}(\eta - y)$. The transformations indicated preserve the scalar product: $(\lambda, \eta - y) = (\mu, z)$,

and the volume element: $d\lambda d\mu = d\eta dz$. Therefore in the new variables (1.9) becomes

$$u(t, x, y) = \frac{1}{2(2\pi)^{\frac{n-1}{2}}} \int_{x-t}^{x+t} d\xi \int_{R^{n-1}} J_0(k|\mu|) d\mu \frac{1}{2(2\pi)^{\frac{n-1}{2}}} \int_{R^{n-1}} h(\xi, y + T'z) e^{-i(\mu, z)} dz. \quad (1.11)$$

Here $|\mu| = \sqrt{\sum_{i=1}^{n-1} \mu_i^2}$; a similar notation for the length of a vector will be used later on. Everywhere in this article the vectors in coordinate form are written as a row even though the algebraic operations over them are performed by the usual rule of actions with column-vectors. For instance, when multiplying vectors by a matrix, we write the matrix on the left. The conventions mentioned, adopted for convenience of writing, will be applied without special stipulation.

The following simplifications of (1.11) are connected with the computation of the integrals occurring in it. Note that the direct change of order of integration with respect to the variables μ and z leads to a divergent integral. We bypass this formal difficulty by noting that the inner integral in (1.11) can be treated as the Fourier transform of the generalized function given by the locally integrable function $J_0(k|\mu|)$. The next section contains material connected with the computation of this Fourier transform.

§ 2 Fourier transform of the Bessel function $J_0(a|\mu|)$

We use some well-known facts from the theory of generalized functions (distributions) [158]. We consider a generalized function defined by the locally integrable function (a is an arbitrary nonnegative constant)

$$J_0(a|\mu|) = J_0 \left(a \sqrt{\sum_{i=1}^{n-1} \mu_i^2} \right)$$

and acting on test functions $\psi(\mu_1, \dots, \mu_{n-1})$ by the formula

$$\langle J_0(a|\mu|), \psi(\mu) \rangle = \int_{R^{n-1}} J_0(a|\mu|) \psi(\mu) d\mu. \quad (2.1)$$

This generalized function is, obviously, slowly growing, so that we can speak of the Fourier transform $\mathcal{F}[J_0(a|\mu|)]$. In this section it is convenient for us

to denote by the symbol \mathcal{F} the Fourier transform of both ordinary as well as generalized functions.

The function

$$\psi(\mu) = \mathcal{F}[\varphi(z)] = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{R^{n-1}} \varphi(z) e^{-i(\mu, z)} dz$$

is the Fourier transform of the function $\varphi(z)$ from the space of rapidly decreasing functions. The inverse Fourier transform is determined by the formula

$$\varphi(z) = \mathcal{F}^{-1}[\psi(\mu)] = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{R^{n-1}} \psi(\mu) e^{i(\mu, z)} d\mu.$$

The Fourier transform $\mathcal{F}[J_0(a|\mu|)]$ of the generalized function $J_0(a|\mu|)$ acts by the formula

$$\langle \mathcal{F}[J_0(a|\mu|)], \varphi(z) \rangle = \langle J_0(a|\mu|), \psi(\mu) \rangle. \quad (2.2)$$

We find the Fourier transform of the generalized function $J_0(a|\mu|)$ for the case of odd $n \geq 3$; $\mu = (\mu_1, \dots, \mu_{n-1})$ with $n = 2m + 3$, $m \geq 0$. This proves to be sufficient for the desired simplification of (1.11).

We introduce the generalized function j_a acting on $\varphi(z_1, \dots, z_{2m+2})$ by the formula

$$\langle j_a(z), \varphi(z) \rangle = \left[\frac{d^m}{ds^m} \left(s^m \int_{\omega_{2m+2}} \varphi(\sqrt{s}\zeta) d\omega \right) \right]_{s=a^2}, \quad (2.3)$$

where ω_{2m+2} is the unit sphere with center at the origin in a $(2m+2)$ -dimensional space, $d\omega$ is a surface element of the sphere, $\zeta \in \omega_{m+2}$, and $z = \sqrt{s}\zeta$.

Theorem 16.1. Let n be odd and $n = 2m + 3$, $m \geq 0$. Then

$$\mathcal{F}[J_0(a|\mu|)] = 2^m j_a.$$

Proof. It is enough to show that

$$\mathcal{F}^{-1}[j_a] = \frac{1}{2^m} J_0(a|\mu|). \quad (2.4)$$

We compute the left-hand side of this equality. We first note that the generalized function j_a has compact support. Therefore

$$\mathcal{F}^{-1}[j_a] = \frac{1}{(2\pi)^{m+1}} \langle j_a(z), e^{i(\mu, z)} \rangle = \frac{1}{(2\pi)^{m+1}} \left[\frac{d^m}{ds^m} \left(s^m \int_{\omega_{2m+2}} e^{i\sqrt{s}(\mu, \zeta)} d\omega \right) \right]_{s=a^2}.$$

We transform the expression obtained by introducing the angle θ ($0 \leq \theta \leq \pi$) between the vectors μ and ζ . We have $(\mu, \zeta) = |\mu| \cos \theta$ and

$$\int_{\omega_{2m+2}} e^{i\sqrt{s}(\mu, \zeta)} d\omega = \Omega_{2m+1} \int_0^\pi e^{i\sqrt{s}|\mu| \cos \theta} (\sin \theta)^{2m} d\theta,$$

where

$$\Omega_{2m+1} = \frac{2\pi^{m+\frac{1}{2}}}{\Gamma(m+\frac{1}{2})}$$

is the surface area of the unit sphere in a $(2m+1)$ -dimensional space. The last integral can be expressed in terms of a Bessel function (see [176], Chapter II, §2.3, formula (2)):

$$\int_0^\pi e^{i\sqrt{s}|\mu| \cos \theta} (\sin \theta)^{2m} d\theta = 2^m \frac{\Gamma(m+\frac{1}{2}) \Gamma(\frac{1}{2})}{(\sqrt{s}|\mu|)^m} J_m(\sqrt{s}|\mu|).$$

Hence

$$\mathcal{F}^{-1}[j_a] = \left[\frac{d^m}{ds^m} \left(s^{\frac{m}{2}} \frac{1}{|\mu|^m} J_m(\sqrt{s}|\mu|) \right) \right]_{s=a^2}.$$

To simplify the right-hand side we introduce the new variable $x = |\mu|\sqrt{s}$. Then $\frac{d}{ds} = \frac{|\mu|^2}{2} \frac{d}{x dx}$ and

$$\mathcal{F}^{-1}[j_a] = \frac{1}{2^m} \left[\left(\frac{d}{x dx} \right)^m (x^m J_m(x)) \right]_{x=a|\mu|}.$$

Using the reduction formula for Bessel functions [176]

$$\left(\frac{d}{x dx} \right)^m [x^\nu J_\nu(x)] = x^{\nu-m} J_{\nu-m}(x),$$

we obtain the required equality (2.4).

§ 3 Representation of the solution

We now return to the calculations connected with the simplification of the formula (1.11) for the solution of Cauchy problem (1.3). Let us begin with odd $n \geq 3$. Setting $n = 2m + 3$ ($m \geq 0$) and using the results of § 2, we write (1.11) in the form

$$u(t, x, y) = \frac{1}{2(2\pi)^{m+1}} \int_{x-t}^{x+t} < J_0(k|\mu|), \mathcal{F}[\varphi_\xi(z)] > d\xi,$$

where $\varphi_\xi(z) = h(\xi, y + T'z)$. According to definition (2.2) of the Fourier transform of generalized functions, we have

$$\langle J_0(k|\mu|), \mathcal{F}[\varphi_\xi(z)] \rangle = \langle \mathcal{F}[J_0(k|\mu|)], \varphi_\xi(z) \rangle,$$

and, therefore,

$$u(t, x, y) = \frac{1}{2(2\pi)^{m+1}} \int_{x-t}^{x+t} \langle \mathcal{F}[J_0(k|\mu|)], \varphi_\xi(z) \rangle d\xi. \quad (3.1)$$

In (3.1) we substitute the value of the Fourier transform $\mathcal{F}[J_0(k|\mu|)]$ of the generalized function $J_0(k|\mu|)$, given by Theorem 2.1, and we make use of the definition (2.3) of the generalized function j_a with nonnegative $a = k = \sqrt{x+t-\xi}$. As a result we obtain the following final representation of the solution of Cauchy problem (1.3) for the case of odd $n = 2m + 3, m \geq 0$:

$$u(t, x, y) = \frac{1}{4\pi^{m+1}} \int_{x-t}^{x+t} \left[\frac{\partial^m}{\partial s^m} \left(s^m \int_{\omega_{2m+2}} h(\xi, y + \sqrt{s} T' \zeta) d\omega \right) \right]_{s=x+t-\xi} d\xi. \quad (3.2)$$

In (3.2) we have used the notation introduced earlier: T' is the transposed matrix of transformation (1.10), and $\zeta = (\zeta_1, \dots, \zeta_{2m+2})$ are the coordinates of points on the unit sphere ω_{2m+2} with center at the origin in a $(2m+2)$ -dimensional space, over which the integration is carried out.

To obtain the formula giving the solution of Cauchy problem (1.3) in the case of even $n \geq 2$ we make use of a descent method. Let $n = 2m+2, m \geq 0$. We consider the following auxiliary Cauchy problem:

$$\begin{aligned} \tilde{L}[u] &\equiv u_{tt} - u_{xx} - \sum_{i,j=1}^{2m+1} a_{ij}(x-t) u_{y_i y_j} - u_{pp} = 0, \\ u|_{t=0} &= 0, \quad u_t|_{t=0} = h(x, y_1, \dots, y_{2m+1}). \end{aligned} \quad (3.3)$$

Since the function h is independent of p , by virtue of the uniqueness of the solution of a Cauchy problem the solution u of problem (3.3) will also be independent of this variable, and by the same token will be the solution of the Cauchy problem we are interested in

$$\begin{aligned} L[u] &\equiv u_{tt} - u_{xx} - \sum_{i,j=1}^{2m+1} a_{ij}(x-t) u_{y_i y_j} = 0, \\ u|_{t=0} &= 0, \quad u_t|_{t=0} = h(x, y), \end{aligned} \quad (3.4)$$

where $y = (y_1, \dots, y_{2m+1})$.

We obtain a formula for the solution of problem (3.4) by taking advantage of the representation (3.2) of the solution of the Cauchy problem with $n = 2m + 3$, written for the auxiliary problem (3.3). Let T be the $(2m + 1) \times (2m + 1)$ matrix of transformation (1.10) corresponding to the problem (3.4) being examined, and let T' be the transposed matrix. Then, as the $(2m + 2) \times (2m + 2)$ matrix of transformation (1.10) corresponding to the auxiliary problem (3.3), we can take the matrix

$$\tilde{T} = \begin{pmatrix} & & & 0 \\ & & & 0 \\ & T & & \vdots \\ & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \quad (3.5)$$

The transposed matrix \tilde{T}' has the same form, with T' instead of T . Further, let $\tilde{\zeta} = (\zeta_1, \dots, \zeta_{2m+1}, \sigma) \in \omega_{2m+2}$. The solution of the auxiliary problem (3.3), according to (3.2), has the form

$$u(t, x, y) = \frac{1}{4\pi^{m+1}} \int_{x-t}^{x+t} \left[\frac{\partial^m}{\partial s^m} \left(s^m \int_{\omega_{2m+2}} h(\xi, y + \sqrt{s} \tilde{T}' \tilde{\zeta}) d\omega \right) \right]_{s=x+t-\xi} d\xi. \quad (3.6)$$

Starting from (3.6), we arrive at the desired result after the following simple calculations. The function h in (3.3) is independent of p , and therefore the integral over the surface of the unit sphere ω_{2m+2} can be written as an integral over a $(2m + 1)$ -dimensional ball. Indeed, let K_{2m+1} be a ball of unit radius with center at the origin in a $(2m + 1)$ -dimensional space and let $\zeta = (\zeta_1, \dots, \zeta_{2m+1}) \in K_{2m+1}$. Then $\tilde{\zeta} = (\zeta, \sigma) \in \omega_{2m+2}$ if σ is real and $|\sigma| \leq 1$. We make use of the special form (3.5) of the matrix \tilde{T} , and we rewrite the integral over ω_{2m+2} in (3.6) in the following form:

$$\int_{\omega_{2m+2}} h(\xi, y + \sqrt{s} \tilde{T}' \tilde{\zeta}) d\omega = 2 \int_{K_{2m+1}} h(\xi, y + \sqrt{s} T' \zeta) \frac{d\zeta}{\sqrt{1 - |\zeta|^2}}. \quad (3.7)$$

Here we used the equality $\tilde{T}' \tilde{\zeta} = (T' \zeta, \sigma)$ and the standard transformation of integrals over the surface of a sphere in the $(2m + 2)$ -dimensional space of functions independent of one of the variables to integrals over a $(2m + 1)$ -dimensional ball.

(3.6) and (3.7) yield the following representation of the solution of Cauchy problem (3.4):

$$u(t, x, y) = \quad (3.8)$$

$$\frac{1}{2\pi^{m+1}} \int_{x-t}^{x+t} \left[\frac{\partial^m}{\partial s^m} \left(s^m \int_{K_{2m+2}} h(\xi, y + \sqrt{s} T' \zeta) \frac{d\zeta}{\sqrt{1 - |\zeta|^2}} \right) \right]_{s=x+t-\xi} d\xi.$$

The results obtained are stated in the following theorem.

Theorem 16.2. The solution of the Cauchy problem,

$$u_{tt} - u_{xx} - \sum_{i,j=1}^{n-1} a_{ij}(x-t) u_{y_i y_j} = 0, \quad t > 0,$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = h(x, y)$$

is given by

$$u(t, x, y) = \quad (3.9)$$

$$\frac{1}{4\pi^{\frac{n-1}{2}}} \int_{x-t}^{x+t} \left[\left(\frac{\partial}{\partial s} \right)^{\frac{n-3}{2}} \left(s^{\frac{n-3}{2}} \int_{\omega_{n-1}} h(\xi, y + \sqrt{s} T' \zeta) d\omega \right) \right]_{s=x+t-\xi} d\xi$$

for odd $n \geq 3$ and

$$u(t, x, y) = \quad (3.10)$$

$$\frac{1}{2\pi^{\frac{n}{2}}} \int_{x-t}^{x+t} \left[\left(\frac{\partial}{\partial s} \right)^{\frac{n-2}{2}} \left(s^{\frac{n-2}{2}} \int_{K_{n-1}} h(\xi, y + \sqrt{s} T' \zeta) \frac{d\zeta}{\sqrt{1 - |\zeta|^2}} \right) \right]_{s=x+t-\xi} d\xi$$

for even $n \geq 2$, where ω_{n-1} is the unit sphere with center at the origin in an $(n-1)$ -dimensional space, K_{n-1} is a ball of unit radius with center at the origin in an $(n-1)$ -dimensional space, the integration variable $\zeta = (\zeta_1, \dots, \zeta_{n-1})$ ranges over the sphere ω_{n-1} and the ball K_{n-1} respectively, and T' denotes the transposed matrix of transformation (1.10).

Remark 16.1. The nonsingular matrix T' is not uniquely defined. The only requirement imposed on T' is the equation

$$A(\xi) - A(x-t) = T' T,$$

where $A(\sigma)$ is an $(n-1) \times (n-1)$ matrix with elements $A_{ij}(\sigma) = \int a_{ij}(\sigma) d\sigma$, $i, j = 1, \dots, n-1$. It is easy to see that this equality defines the matrix T to within multiplication by an arbitrary orthogonal matrix. In particular, as T we can take the symmetric positive definite matrix $T = [A(\xi) - A(x-t)]^{1/2}$, infinitely differentiable in the variables x , t , and $\xi > x-t$.

§ 4 The Huygens principle

In this section we discuss a special property of equation (1.1), connected with the fulfillment of the so-called Huygens principle. Historically the Huygens principle arose in connection with problems in optics. At first the Huygens principle meant only a certain method for constructing a wave front. Later on the Huygens principle was also said to be a statement on the existence of the trailing edge of a light wave caused by a light source localized in space and time. Here, by the Huygens principle we shall mean precisely this statement, specifically its formulation given by Hadamard [61].

Hadamard's starting point was the investigations of Kirchhoff, Beltrami, and Volterra, devoted to ascertaining the mathematical meaning of the Huygens principle for the classical wave equation in three-dimensional space

$$u_{tt} - \Delta u = 0. \quad (4.1)$$

As a result of these investigations it was established that the Huygens principle expresses a specific property of the solution of the Cauchy problem for equation (4.1). The property mentioned is that the solution of the Cauchy problem at the point (t, x_0, y_0, z_0) depends upon the values of the initial data and of their derivatives only on a sphere of radius t with center at the point (x_0, y_0, z_0) in the space of the variables x, y, z . In other words, the solution of the Cauchy problem for equation (4.1) is determined by the values of the initial data and of their derivatives only on the intersection of the inverse characteristic cone with the manifold where the initial data is given. The validity of this statement follows from the explicit formula for the solution.

Tedone obtained the formulas giving the solution of the wave equation

$$u_{tt} - \sum_{i=1}^n u_{x_i x_i} = 0. \quad (4.2)$$

with an arbitrary number of space variables. From his formulas it followed that the Huygens principle, in the formulation presented above, is fulfilled for equation (4.2) with an arbitrary odd number $n \geq 3$ of space variables and is not fulfilled if the number of space variables is even.

Hadamard formulated the Huygens principle for arbitrary linear second-order hyperbolic equations and obtained a number of important results connected with the description of the class of equations satisfying the Huygens principle. Since the problem of a complete description of all such equations proved to be difficult and has not yet been solved, the construction of appropriate examples is important. A number of papers were devoted to this question by Hadamard himself as well as by other authors.

Below we discuss the examples known, and we consider equation (1.1) from this point of view. Now, however, we wish to show that in the case of equation (4.2) Theorem 16.2 yields the known formulas for the solution of the Cauchy problem in spite of the formal difference between the formulas being compared. Thus we shall have established that the representation of the solution of the Cauchy problem for equation (1.1), given by Theorem 16.2, has, essentially, as simple a structure as the known formulas [35] in the case of the wave equation (4.2). For our purpose it is sufficient to consider only odd values of $n \geq 3$. In this case we once again take $n = 2m + 3$ with nonnegative integers m , and we use the notation (3.2) for formula (3.9).

For equation (4.2) we consider the Cauchy problem with initial data

$$u|_{t=0} = 0, \quad u_t|_{t=0} = h(x), \quad (4.3)$$

where $x = (x_1, \dots, x_{2m+3})$. The classical representation of the solution of problem (4.2), (4.3) (see [35]) has, in our notation, the form

$$u(t, x) = \frac{1}{4\pi^{m+1}} \left(\frac{\partial}{\partial t^2} \right)^m \left[t^{2m+1} \int_{\omega_{2m+3}} h(x + t\alpha) d\omega \right], \quad (4.4)$$

where $\alpha \in \omega_{2m+3}$ is the integration variable. Formula (4.4) is easily put into the form (3.2) by the following transformations. We rename the independent variables: $x = x_1$, $y_i = x_{i+1}$ ($i = 1, \dots, 2m+2$), and in (4.4) we change the variables of integration by introducing the variables ξ and ζ_i ($i = 1, 2, \dots, 2m+2$) in accordance with the formulas

$$\alpha_1 = \frac{\xi - x}{t}, \quad \alpha_{i+1} = \frac{\sqrt{t^2 - (\xi - x)^2}}{t} \zeta_i \quad (i = 1, \dots, 2m+2).$$

Then $x - t \leq \xi \leq x + t$ and $|\zeta| = 1$, where $\zeta = (\zeta_1, \dots, \zeta_{2m+2})$. After the indicated changes of variables equation (4.4) is rewritten as

$$u(t, x, y) = \frac{1}{4\pi^{m+1}} \left(\frac{\partial}{\partial t^2} \right)^m \int_{x-t}^{x+t} d\xi \int_{\omega_{2m+2}} [t^2 - (\xi - x)^2]^m h(\xi, y + \sqrt{t^2 - (\xi - x)^2} \zeta) d\omega$$

with $y = (y_1, \dots, y_{2m+2})$. We have used here the connection between the area elements $d\omega_{2m+2}$ and $d\omega_{2m+3}$ of the surfaces of the unit sphere in spaces of dimensions $2m+2$ and $2m+3$, respectively; in our notation

$$d\omega_{2m+3} = \frac{[t^2 - (\xi - x)^2]^m}{t^{2m+1}} d\omega_{2m+2} d\xi.$$

The expression $[t^2 - (\xi - x)^2]^m$ vanishes at $\xi = x - t$ and $\xi = x + t$ together with all of derivatives with respect to t^2 up to order $m - 1$, inclusively. Therefore

$$u(t, x, y) = \frac{1}{4\pi^{m+1}} \int_{x-t}^{x+t} d\xi \int_{\omega_{2m+2}} \left(\frac{\partial}{\partial t^2} \right)^m \{ [t^2 - (\xi - x)^2]^m h(\xi, y + \sqrt{t^2 - (\xi - x)^2} \zeta) \} d\omega.$$

We now prove the identity

$$\begin{aligned} & \left(\frac{\partial}{\partial t^2} \right)^m \{ [t^2 - (\xi - x)^2]^m h(\xi, y + \sqrt{t^2 - (\xi - x)^2} \zeta) \} \\ &= \left[\left(\frac{\partial}{\partial s} \right)^m \{ s^m h(\xi, y + \sqrt{s(\xi - x + t)} \zeta) \} \right]_{s=x+t-\xi}, \end{aligned} \quad (4.5)$$

and hence complete our transformation of (4.4) to the form (3.2). If we make a change of variable by introducing the variable $\tau = t^2 - (\xi - x)^2$ instead of t^2 , then the left-hand side of (4.5) becomes

$$\left[\left(\frac{\partial}{\partial \tau} \right)^m \{ \tau^m h(\xi, y + \sqrt{\tau} \zeta) \} \right]_{\tau=t^2-(\xi-x)^2}.$$

By changing the variable s to the variable $\sigma = s(\xi - x + t)$, the right-hand side of (4.5) is transformed to

$$\left[\left(\frac{\partial}{\partial \sigma} \right)^m \{ \sigma^m h(\xi, y + \sqrt{\sigma} \zeta) \} \right]_{\sigma=t^2-(\xi-x)^2}.$$

The expressions obtained coincide, and (4.5) is proved.

We now return to the discussion of the Huygens principle for general linear second-order hyperbolic equations. Following Hadamard [61], by the Huygens principle we mean the following statement. For any point x_0 the value $u(x_0)$ of the solution $u(x)$ of the Cauchy problem for the equation

$$\sum_{i,j=0}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=0}^n b_i(x) u_{x_i} + c(x) u = 0 \quad (4.6)$$

with arbitrary data on a surface S of spatial type depends on the values of the initial data and of their derivatives up to some order only on the intersection of S and of the inverse characteristic conoid with vertex at x_0 .

As we said above, the classical example of an equation satisfying the Huygens principle is the wave equation with an odd number of space variables. This example was the only one known for a long time. This circumstance is partly explained by the theorem proved by Mathisson [128]

for equations with three space variables. He proved that if in (4.6) with $n = 3$ all the coefficients a_{ij} are constant, then the only equation satisfying the Huygens principle is the wave equation (4.1). Speaking more precisely, any such equation can be reduced to the wave equation by means of certain equivalence transformations which we shall take up in the next section.

An example of an equation satisfying the Huygens principle and not equivalent to the wave equation was first constructed by Stellmacher. He examined equation (4.6) with $n = 5$, $b_i = 0$ and $a_{ij} = \text{const.}$ ($i, j = 0, 1, \dots, 5$), and described the form of the coefficient $c(x)$, starting from the requirement of fulfillment of the Huygens principle. The equation he found

$$u_{tt} - \sum_{i=1}^5 u_{x_i x_i} - \frac{2}{t^2} u = 0$$

is not equivalent to the wave equation and satisfies the Huygens principle, as is seen from the explicit formula Stellmacher obtained for the solution of the Cauchy problem. Similar examples of equations where a principal part is that of the wave equation were constructed later by him also for other odd n greater than five. As Mathisson's theorem shows, the restriction $n \geq 5$ is essential when constructing these examples.

In the case of $n = 3$ an example of an equation of form (4.6) satisfying the Huygens principle and not equivalent to the wave equation was presented by Günther [60] and by us [92]. This example is a four-dimensional variant of equation (1.1). Günther established the validity of the Huygens principle with the aid of Hadamard's criterion. In our paper that fact followed from the explicit formula for the solution of the Cauchy problem.

Let us discuss this example, following its presentation in [79]. We consider the equation

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0, \quad (4.7)$$

and we solve the Cauchy problem with the data

$$u|_{t=0} = 0, \quad u_t|_{t=0} = h(x, y, z). \quad (4.8)$$

Its solution has the form

$$u(t, x, y, z) = \frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_0^{2\pi} h(\xi, y + A \cos \theta, z + B \cos \theta + C \sin \theta) d\theta \quad (4.9)$$

where

$$A = \sqrt{(x+t-\xi)[F(\xi) - F(x-t)]}, \quad B = \frac{x+t-\xi}{A} [\Phi(\xi) - \Phi(x-t)],$$

$$C = \sqrt{t^2 - (\xi - x)^2 - B^2},$$

while F and Φ are the primitives of functions f and φ , respectively. (4.9) is obtained from (3.9) with $n = 3$ if as T' we choose the matrix

$$T' = \begin{pmatrix} \frac{A}{\sqrt{x+t-\xi}} & 0 \\ \frac{B}{\sqrt{x+t-\xi}} & \frac{C}{\sqrt{x+t-\xi}} \end{pmatrix}.$$

For equation (4.7) we can construct the characteristic conoid in explicit form. Let the point (t^0, x^0, y^0, z^0) be fixed. The characteristic conoid with vertex at this point is given by the equation

$$\begin{aligned} (t - t^0)^2 &= (x - x^0)^2 \\ &+ \frac{(x - t) - (x^0 - t^0)}{[(x - t) - (x^0 - t^0)][F(x - t) - F(x^0 - t^0)] - [\Phi(x - t) - \Phi(x^0 - t^0)]^2} \\ &\times \left\{ [(x - t) - (x^0 - t^0)](y - y^0)^2 - 2[\Phi(x - t) - \Phi(x^0 - t^0)](y - y^0)(z - z^0) \right. \\ &\quad \left. + [F(x - t) - F(x^0 - t^0)](z - z^0)^2 \right\}. \end{aligned} \quad (4.10)$$

From (4.9) we see that the solution of problem (4.7), (4.8) is expressed as an integral of the initial function h over a two-dimensional surface in the space of variables x, y, z , being the intersection of the characteristic conoid (4.10) with the initial manifold $t = 0$. This signifies the validity of the Huygens principle for equation (4.7).

Let us now consider the question of the validity of the Huygens principle for equation (1.1) with an arbitrary number of independent variables. Theorem 16.2 essentially answers this question. Indeed, if $n \geq 3$ is odd, then the solution of Cauchy problem (1.3), given by (3.9), is determined by the values of the initial function h and of its derivatives on an $(n - 1)$ -dimensional manifold in the n -dimensional space of variables x and y . From Lemma 16.1 it follows that the solution of the general Cauchy problem (1.2) also is determined by the values of the initial data f and g and of their derivatives up to some order on the same $(n - 1)$ -dimensional manifold. It follows from Hadamard's results [61] that this manifold should be the intersection of the inverse characteristic conoid and the hyperplane $t = 0$ where the initial data is given. The Huygens principle is connected only with the properties of the equation itself, and its validity does not depend upon the form of the initial manifold. Therefore, we can state that for odd $n \geq 3$ equation (1.1) satisfies the Huygens principle. As Hadamard showed, equations of form

(4.6) satisfying the Huygens principle do not exist for even n or for $n = 1$. In the case of equation (1.1) we see this directly from (3.10).

We summarize the properties of equation (1.1) in the following theorem.

Theorem 16.3. The Huygens principle is valid for the linear hyperbolic second-order equation

$$u_{tt} - u_{xx} - \sum_{i,j=1}^{n-1} a_{ij}(x-t)u_{y_i y_j} = 0$$

with arbitrary smooth coefficients a_{ij} ($a_{ij} = a_{ji}$) if $n \geq 3$ is odd, and is not valid for the remaining n .

Thus, equation (16.1) with odd $n \geq 3$ provides an example of an equation for which the Huygens principle is valid for all values of n mentioned. For arbitrary coefficients $a_{ij}(x-t)$ this equation does not reduce either to the wave equation (4.2) or to Stellmacher's examples. This will be proved in the next section, which is devoted to an analysis of equivalence transformations of linear second-order differential equations.

§ 5 Equivalence transformation

Here we shall use certain results of classical Riemannian geometry. Accordingly, we shall employ upper and lower indices for vectors and tensors, the summation convention for repeated indices, and other commonly adopted notation of local Riemannian geometry [46]. The partial derivatives u_{x_i} and $u_{x_i x_j}$ will be abbreviated to u_i and u_{ij} , respectively. We write an arbitrary linear homogeneous second-order equation as

$$P(u) \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0, \quad (5.1)$$

where $x = (x^0, \dots, x^n)$ and the summation over i and j is taken from 0 to n . It is convenient to connect an $(n+1)$ -dimensional Riemannian space V_{n+1} (see [61]) with equation (5.1), whose leading coefficients form a nonsingular symmetric matrix $||a^{ij}||$. The metric tensor a_{ij} of the space V_{n+1} is determined by the matrix $||a_{ij}||$ inverse to $||a^{ij}||$, i.e. $a_{ij}a^{jk} = \delta_i^k$, where δ_i^k is the Kronecker symbol.

Transformations obtained by superposition of the following three basic transformations:

- (a) the nonsingular change of independent variables $\bar{x}^i = \alpha^i(x)$,
 $i = 0, 1, \dots, n$;

(b) the linear transformation of the dependent variable $\bar{u} = \beta(x)u$, $\beta(x) \neq 0$;

(c) the multiplication of the left-hand side of the equation by a nonzero factor $\bar{P}[u] = \gamma(x)P[u]$, $\gamma(x) \neq 0$,
are called the *equivalence transformations* for equation (5.1). Equivalence transformations are characterized by the fact that they do not violate the linearity and homogeneity of the equation.

Two equations of form (5.1) are said to be *equivalent* if one of them can be transformed into the other by some equivalence transformation. Here we shall examine only equations (5.1) of hyperbolic type. It is clear that equivalent equations simultaneously satisfy or do not satisfy the Huygens principle. Therefore examples of equations satisfying the Huygens principle should be considered to be different if they are nonequivalent. In this regard we are interested in the following two questions:

- 1) Is equation (1.1) equivalent to the wave equation (4.2)?
- 2) Is equation (1.1) equivalent to one of Stellmacher's examples?

Equations where a principal part is that of the wave equation were examined in Stellmacher's examples. This enables us to answer both of the above questions using one and the same arguments. To do this we first note that the transformation (a) is simply a transition to new coordinates in V_{n+1} . Transformations (b) and (c) change equation (5.1) in such a way that all the leading coefficients a^{ij} of this equation are multiplied by some nonzero factor. The form of the lower coefficients of (5.1) and the nature of their variation under the equivalence transformations turn out to be immaterial for our arguments.

The remarks made can be formulated geometrically thus. If two equations $P[u] = 0$ and $\tilde{P}[u] = 0$ of form (5.1) are equivalent, then the Riemannian spaces V_{n+1} and \tilde{V}_{n+1} corresponding to them are conformal to each other. Therefore, to answer the questions interesting us it is sufficient to ascertain whether the space V_{n+1} corresponding to equation (1.1) is conformal to a flat space, i.e. to the $(n+1)$ -dimensional Riemannian space corresponding to the wave equation (4.2). We show that the Riemannian space corresponding to equation (1.1) with arbitrary coefficients $a_{ij}(x-t)$ is not conformal to a flat space.

It is sufficient to find some equation of the form (1.1) to which there corresponds a Riemannian space not conformal to a flat space. We shall seek such an equation among the equations of the following simple form:

$$u_{tt} - u_{xx} - \sum_{i,j=1}^{n-2} u_{y_i y_j} - f(x-t)u_{y_{n-1} y_{n-1}} = 0 \quad (5.2)$$

with a positive function f . We write (5.2) in the form (5.1), using the nota-

tion: $x^0 = t$, $x^1 = x$, $x^2 = y_1$, ..., $x^n = y_{n-1}$. Then (5.2) takes the required form (5.1) with coefficients

$$\begin{aligned} b^i &= 0 \quad (i = 0, \dots, n), \quad c = 0, \quad a^{00} = -1, \quad a^{ii} = -1 \quad (5.3) \\ (i &= 1, \dots, n-1), \quad a^{nn} = -f(x^1 - x^0), \quad a^{ij} = 0 \quad \text{for all } i \neq j. \end{aligned}$$

Recall the well-known test for a Riemannian space V_m to be conformal to a flat space [46]. Namely, a space V_m is conformal to an m -dimensional flat space if and only if the Weyl tensor (the tensor of conformal curvature) C_{ijk}^l of space V_m equals zero. This tensor is defined by

$$\begin{aligned} C_{ijk}^l &= R_{ijk}^l + \frac{1}{m-2} (\delta_j^l R_{ik} - \delta_k^l R_{ij} + g_{ik} R_j^l - g_{ij} R_k^l) \quad (5.4) \\ &+ \frac{R}{(m-1)(m-2)} (\delta_k^l g_{ij} - \delta_j^l g_{ik}). \end{aligned}$$

Here g_{jk} is the metric tensor of V_m , while R_{ijk}^l , R_{ik} , and R are the Riemann tensor, the Ricci tensor, and the scalar curvature of this space, respectively. In accordance with the common tensor notation, the mixed tensor R_j^l is obtained from the Ricci tensor R_{ij} by raising the subscript.

We also find functions $f(x-t)$ for which a Riemannian space V_{n+1} with the nonzero Weyl tensor corresponds to equation (5.2). To do this it is sufficient to find a function f with which at least one of the components of the tensor C_{ijk}^l being examined is nonzero.

The metric tensor of V_{n+1} corresponding to (5.2) has, according to (5.3), the following components:

$$a_{00} = 1, \quad a_{ii} = -1, \quad (i = 1, \dots, n-1), \quad a_{nn} = -\frac{1}{f}, \quad a_{ij} = 0 \quad (5.5)$$

for all $i \neq j$. From (5.4), invoking (5.5), we compute the component

$$C_{n1n}^1 = R_{n1n}^1 + \frac{1}{n-1} \left(R_{nn} - \frac{1}{f} R_1^1 \right) + \frac{R}{n(n-1)} \frac{1}{f}. \quad (5.6)$$

From the general formulas we first find expressions for R , R_1^1 , R_{nn} and R_{n1n}^1 . To do this we require the values of the Christoffel symbols computed for the metric tensor (5.5). In the case being examined, from the formulas

$$\Gamma_{jk}^i = \frac{1}{2} a^{il} \left(\frac{\partial a_{lj}}{\partial x^k} + \frac{\partial a_{lk}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^l} \right)$$

it is easy to obtain the following nonzero Christoffel symbols:

$$\Gamma_{nn}^0 = \Gamma_{nn}^1 = -\frac{1}{2} \left(\frac{1}{f} \right)', \quad \Gamma_{1n}^n = \Gamma_{n1}^n = -\Gamma_{0n}^n = -\Gamma_{n0}^n = \frac{1}{2} f \left(\frac{1}{f} \right)'; \quad (5.7)$$

the prime denotes differentiation with respect to the argument. Then

$$R_{ii} = 0 \quad (i = 1, \dots, n), \quad (5.8)$$

$$R_{00} = R_{11} = -R_1^1 = \frac{1}{2} \left[f \left(\frac{1}{f} \right)' \right]' + \frac{1}{4} f^2 \left[\left(\frac{1}{f} \right)' \right]^2,$$

$$R = a^{ij} R_{ij} = R_{00} - \sum_{i=0}^{n-1} R_{ii} - f R_{nn} = 0$$

and

$$R_{n1n}^1 = -\frac{1}{2} \left(\frac{1}{f} \right)'' + \frac{1}{4} f \left[\left(\frac{1}{f} \right)' \right]^2. \quad (5.9)$$

Equations (5.8) and (5.9) yield $R_{n1n}^1 = \frac{1}{f} R_1^1$, and hence (5.6) gives

$$C_{n1n}^1 = \frac{n-2}{n-1} \frac{R_1^1}{f}.$$

We can now find all such positive functions f for which $C_{n1n}^1 = 0$. We introduce the function $h = 1/f$ and we write the value of R_1^1 from (5.8) in the form $R_1^1 = -(\ln \sqrt{h})'' - [(\ln \sqrt{h})']^2$. Then the equality $C_{n1n}^1 = 0$ reduces to the following equation for the function h of one variable $\sigma = x - t$: we have $(\ln \sqrt{h})'' + [(\ln \sqrt{h})']^2 = 0$ with the general solution $h(\sigma) = (a\sigma + b)^2$, where a and b are arbitrary constants.

Thus, $C_{n1n}^1 = 0$ if and only if

$$f(x - t) = [a(x - t) + b]^{-2}. \quad (5.10)$$

Therefore the Weyl tensor of the space V_{n+1} with metric tensor (5.5) does not equal zero for any function f other than a function of form (5.10), and consequently equation (5.2) with an arbitrary function $f(x - t)$ does not reduce either to the wave equation or to Stellmacher's examples. Moreover, equation (1.1) with arbitrary coefficients $a_{ij}(x - t)$ is not equivalent either to the wave equation or to Stellmacher's examples.

Remark 16.2. Equation (1.1) with $n = 3$ is characterized as the unique conformally invariant equation in the Riemannian space V_4 with a nontrivial conformal group [79]. In the case of an arbitrary n the Riemannian space V_{n+1} corresponding to (1.1) also has a nontrivial conformal group, while (1.1) itself is a conformally invariant equation in this space.

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Paper 17

Bianchi-Lie, Bäcklund, and Lie-Bäcklund transformations

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ABSTRACT

Bianchi-Lie and Bäcklund transformations are discussed in the context of two questions posed by Lie which naturally identify two related directions of development of the classical treatment of surface transformations. These transformations are then related to a general structure consisting of Lie-Bäcklund tangent transformations.

Introduction

Recently, it has been shown that by wedding Bäcklund’s idea of infinite-order tangent transformations with Lie’s notion of continuous groups of transformations it is possible to construct a generalization of Lie’s theory of first-order tangent (contact) transformations and its application to differential equations. This generalization is founded upon invertible infinite-order tangent transformations or what we call Lie-Bäcklund tangent transformations acting in a necessarily infinite-dimensional space. Its exposition relies heavily on the calculational techniques of L. V. Ovsiannikov [138], [144] and is discussed elsewhere (Ibragimov and Anderson [91]). Here we relate these transformations to the classical treatment of surface transformations.

The classical treatment of surface transformations is founded upon the idea of higher-order tangent transformations. The original papers of

S. Lie [108], [110] and A.V. Bäcklund [12], [13], [14], [15] which develop this treatment evidence the mutual influence each had on the other's contribution. The basic ideas underlying the possibility and importance of applying higher-order tangent transformations to differential equations were clearly formulated by Lie (see [108], Note on p. 223) as two questions in a 1874 paper. These questions naturally characterize two directions of development evidenced in the classical literature. The first direction was a search for invertible higher-order tangent transformations which was abortive in the classical literature. This direction of development is summarized in Section § 1. The second direction is closely related to the first one but distinct in character. It led to the development of a special type of surface transformation first discovered by Lie [110] in his analytical treatment of Bianchi's [21] construction for surfaces of constant curvature. A formal generalization of type of transformation is due to Bäcklund [13] and these transformations were later called Bäcklund transformations. In this paper these transformations are discussed in some detail in Sections § 2–§ 4 and we conclude Section § 4 by relating these surface transformations to Lie-Bäcklund tangent transformations.

§ 1 Lie's first question

Lie realized that a generalization of the concept of the tangent transformation could be important in applications to second and higher-order differential equations. He formulated this idea in his 1874 paper as the first of two questions (Lie [108], p. 223).

Lie's first question: Are there transformations which are not first-order tangent transformations and for which tangency of higher-order is an invariant condition?

If this question could be answered in the affirmative, one would have a higher-order transformation which is not a prolonged Lie tangent transformation, but which converts any k th-order partial differential equation ($k > 1$) into another one (without raising the order).

Lie predicted a negative answer to this question. From Lie's group theoretical treatment of transformations, it is natural here to understand transformations as invertible maps. With this understanding of transformations as invertible ones, or in classical terminology, as single-valued surface transformations, one can interpret the main results in Bäcklund's first papers (Bäcklund [12], [13]) as a verification of Lie's conjecture. Bäcklund's results can be summarized in the following statement: There are no nontrivial higher-order or infinite-order generalizations of Lie tangent transformations

if one understands a transformation as an invertible map or as a single-valued surface transformation acting invariantly in a finite-dimensional space.

Recently, we have realized this search for invertible higher-order tangent transformations with the following notion of a Lie-Bäcklund tangent transformation. Consider the infinite-dimensional space of variables $x = (x^1, \dots, x^n) \in R^n$, $u \in R$, $u = \{u_{i_1 \dots i_s} | i_1, \dots, i_s = 1, \dots, n\}$, $s = 1, 2, \dots$, and a transformation T :

$$\begin{aligned} x &= f(x, u, u_{(1)}, \dots), \\ u &= \phi(x, u, u_{(1)}, \dots), \\ u_{(1)} &= \psi(x, u, u_{(1)}, \dots), \\ &\dots\dots\dots \end{aligned} \tag{1.1}$$

In equation (1.1), the number of arguments of each of the functions f^i , ϕ is a priori arbitrary and may be finite or infinite. The number of equations in (1.1) is assumed to be infinite. The transformation (1.1) and its extension to the differentials dx , du , $du_{(1)}$, ... represent the prolonged transformation \tilde{T} . Now we introduce the following definition.

Definition 17.1. The transformation T is called a Lie-Bäcklund tangent transformation (∞ -order tangent transformation) if the ∞ -order tangency conditions

$$du - u_j dx^j = 0, \quad du_i - u_{ij} dx^j = 0, \quad \dots, \tag{1.2}$$

are invariant with respect to the action of the prolonged transformation \tilde{T} .

We refer the interested reader to Anderson and Ibragimov [7] for a more detailed exposition of the material presented in this section. With this as a background we now turn to the second direction of development which is the principal focus of this paper.

§ 2 Lie's second question

Lie [110] and Bäcklund [14] showed in later papers that one way to realize the substantive generalization they sought is to consider many-valued surface transformations. The importance of the existence of such transformations, was emphasized by Lie in his second question (Lie [108], p. 223).

Lie's second question: Given a higher-order partial differential equation, does this equation admit a transformation which is not a first-order tangent transformation?

Lie predicted an affirmative answer to this question. Further he recognized that the realization of this possibility would open an important area of investigation (Lie [110], [111]).

Lie's analytical treatment of Bianchi's geometrical construction of a transformation of surfaces of constant curvature was the example which first clearly demonstrated the potential inherent in the notion of many-valued surface transformations. This example also directly leads to what is called in the literature Bäcklund transformations and clarifies their structure. We now proceed to discuss these transformations.

§ 3 Bianchi-Lie transformation

Here we outline Bianchi's geometrical construction. In three-dimensional Euclidean space, consider a surface S of constant negative curvature $-\frac{1}{a^2}$ where the constant $a > 0$ and another surface S' which is related to S in the following way. For every point $M \in S$ there corresponds a point $M' \in S'$ such that:

- (i) $|MM'| = a$, where $|MM'|$ is the length of the line segment MM' ;
- if τ, τ' are tangent planes to S, S' at M, M' respectively, then
- (ii) $MM' \in \tau$;
- (iii) $MM' \in \tau'$;
- (iv) $\tau \perp \tau'$.

Bianchi [21] demonstrated that S' is also a surface of the same constant curvature $-\frac{1}{a^2}$.

In order to clarify the analytical structure of this transformation and investigate the properties of the family of transformed surfaces S' , Lie expressed Bianchi's geometrical construction in an equivalent analytical form.

Given a surface S of constant curvature $-\frac{1}{a^2}$ the surface S satisfies the second-order partial differential equation

$$s^2 - rt = \frac{1 + p^2 + q^2}{a^2} \quad (3.1)$$

where the usual notation p, q, r, s, t is employed for the first and second derivatives. If (X, Y, Z, P, Q) is a surface element of the transformed surface S' , then conditions (i)-(iv) in Bianchi's construction can be expressed

in the following form which we call the Bianchi-Lie transformation:

$$\begin{aligned}
 (x - X)^2 + (y - Y)^2 + (z - Z)^2 &= a^2, \\
 p(x - X) + q(y - Y) - (z - Z) &= 0, \\
 P(x - X) + Q(y - Y) - (z - Z) &= 0, \\
 pP + qQ + 1 &= 0.
 \end{aligned} \tag{3.2}$$

First we observe, that given any surface element (x, y, z, p, q) equation (3.2) gives four relations between the five quantities X, Y, Z, P, Q ; hence there is a one-fold infinity of surface elements (X, Y, Z, P, Q) satisfying (3.2).

In order to treat (3.2) as a surface transformation, consider z as a given function of x, y and regard p, q, r, s, t as the usual first and second-order derivatives of this function. Let S denote the surface described by the function $z = z(x, y)$. Then, we seek the condition that any element (X, Y, Z, P, Q) obtained by means of (3.2) from a surface element (x, y, z, p, q) of the surface S is also a surface element, i.e.,

$$\frac{\partial P}{\partial Y} - \frac{\partial Q}{\partial X} = 0 \quad \text{on } S. \tag{3.3}$$

The following theorem summarizes Lie's results for transformation (3.2) of surfaces of constant curvature.

Theorem 17.1. (Lie [110].) Equation (3.1) admits transformation (3.2) and transformation (3.2) is defined only on solutions of (3.1).

According to Theorem 17.1 and the structure of the transformation (3.2) one can construct by quadratures a family of surfaces of constant curvature starting from some given one. With this example Lie demonstrated that many-valued surface transformations can be used in a manner similar to that in his previous theory of differential equations based on first-order tangent and point transformations. In particular, equation (3.2), regarded as a transformation, is admitted by the differential equation (3.1) and as a direct consequence of the definition of invariance one has the property that such a transformation converts a solution of equation (3.1) into a solution(s) of the same equation. For a given solution $z = z(x, y)$ of (3.1) the corresponding transformed family of surfaces is found by quadratures, namely, here as a solution of a completely integrable system of two first-order partial differential equations for the unknown function $Z(X, Y)$, which is obtained from (3.2) when one substitutes $z = z(x, y)$ into (3.2) and then uses two of these equations to eliminate the variables x and y . This procedure applied to

four general equations relating (x, y, z, p, q) , (X, Y, Z, P, Q) instead of (3.1) leads to transformations which are called Bäcklund transformations in the literature.

§ 4 Bäcklund transformations

Bäcklund [15] geometrically generalized Bianchi's result to surfaces S, S' that are related by a construction which is obtained from Bianchi's construction by replacing the condition of orthogonality of the two tangent planes τ, τ' with the condition that the angle between these tangent planes is fixed.

Darboux [37] completed the geometrical analysis of Bianchi's construction by replacing Bianchi's conditions (ii), (iii) with the conditions that the line segment MM' makes fixed angles (not necessarily equal to each other) with the tangent planes τ, τ' and retaining Bäcklund's modification. He classified the cases for which the analog of (3.2) is a surface transformation. In particular, he recovered Bäcklund's generalization of the Bianchi-Lie transformation as a subcase. Further, although Bäcklund's construction (called in the classical geometrical literature (Darboux [37]; Bianchi [23]) a Bäcklund transformation) geometrically generalized Bianchi's construction, analytically it turned out to be only a simple composition of a Bianchi-Lie transformation and a one-parameter Lie group of dilatations. A nontrivial analytical generalization of Bianchi-Lie transformation was introduced by Bäcklund [14] who considered two sets of surface elements:

$$F_i(x, y, z, p, q, X, Y, Z, P, Q) = 0, \quad i = 1, 2, 3, 4. \quad (4.1)$$

A literal repetition of Lie's considerations and techniques for treating (3.2) applied to (4.1) leads to what is called in the literature a Bäcklund transformation (in the analytical sense as contrasted with the prior geometrical sense) (Clairin [33], Goursat [57], Ames [3]). In particular, we recall that given equation (3.2), Lie posed and solved the problem of determining the family of surfaces in (x, y, z) -space such that (3.2) acts on the member of this family as a surface transformation in the previously described sense. His result as expressed by Theorem 17.1 is that this family is a family of surfaces of constant curvature. Now turning to equation (4.1), and following Lie, we see that substituting a given function $z = z(x, y)$ in (4.1), and using two of the resulting relations to eliminate x, y the two remaining relationships implied by (4.1) represent an over-determined system of two first-order partial differential equations in one unknown function which we

denote as

$$\begin{aligned} f(X, Y, Z, P, Q) &= 0, \\ \phi(X, Y, Z, P, Q) &= 0. \end{aligned} \tag{4.2}$$

The consistency conditions for this system have the form of partial differential equations for the function $z(x, y)$ and are stated in their general form in Bäcklund's paper ([14], p. 311). If $z(x, y)$ satisfies these consistency conditions, then equation (4.1) is considered as a transformation of the surface $z = z(x, y)$ in (x, y, z) -space into a surface in (X, Y, Z) -space and this latter surface is given as a solution surface of the system (4.2) now treated as an integrable system.

Lie's treatment of Bianchi's construction when extended to equation (4.1) leads to the consideration of several subtypes of surface transformations all christened Bäcklund transformations in the literature. The variety of subtypes of these transformations is connected with the possibility of posing different problems when one treats (4.1) as a surface transformation. The sources of these possibilities lie principally in applications.

Analyzing what is common to all these subtypes of transformations, one can define a Bäcklund transformation as one that is specified by an overdetermined system of first-order differential equations of the form (4.1). As Goursat [57] has remarked one can generalize this form in many ways, including increasing the dimension of the underlying space(s), the order of the surface elements, and the number of relations in (4.1), etc.

We conclude this section with the observation that the Bäcklund transformations of the form (4.1) can be related to Lie-Bäcklund tangent transformations as defined in § 1 in the following way. Here we confine ourselves to a heuristic argument and without loss of generality we take

$$\begin{aligned} X &= x, \\ Y &= y, \\ P &= \psi_1(x, y, z, p, q, Z), \\ Q &= \psi_2(x, y, z, p, q, Z), \end{aligned} \tag{4.3}$$

Now, we suppose that (4.3) is a given Bäcklund transformation of a family of surfaces M . Any surface $z = z(x, y)$ in (x, y, z) -space is specified at any fixed (x, y) in point language by the values of z and its derivatives p, q, r, s, t, \dots at (x, y) (Bäcklund [12]). Now if the surface $z = h(x, y)$ belongs to the family M , equations (4.3) can be integrated. Let $Z = Z_h(x, y)$ be a particular solution for a given h , then if we take $z = h(x, y)$ to be an arbitrary element of M and specifying any surface by the element $(x, y, z, p, q, r, s, t, \dots)$, then the solution $Z_h(x, y)$ becomes $Z = \phi(x, y, z, p, q, r, s, t, \dots)$. This formula when added to (4.3) and extended by differentiation and elimination

yields a Lie-Bäcklund tangent transformation (Definition 1.1).

$$\begin{aligned} X &= x, & Y &= y, \\ Z &= \phi(x, y, z, p, q, r, s, t, \dots), \\ P &= \psi_1(x, y, z, p, q, \phi(x, y, z, p, q, r, s, t, \dots)), \\ Q &= \psi_2(x, y, z, p, q, \phi(x, y, z, p, q, r, s, t, \dots)), \\ &\dots \end{aligned} \tag{4.4}$$

This relation between Bäcklund and Lie-Bäcklund transformations can be strengthened within the context of the application of these transformations to differential equations. (The following results are taken from Fokas and Anderson [51]). In order to discuss this relation it is more convenient to return to the notation of Section § 1. Consider a Bäcklund transformation which maps an n th-order surface element $(x^1, x^2, v, v_{(1)}, \dots, v_{(n)})$ into a family of n th-order surface elements $(x^1, x^2, u, u_{(1)}, \dots, u_{(n)})$. Without loss of generality, we may take it in the form

$$\begin{aligned} u_i &= \psi_i(x, u, v, v_{(1)}, \dots, v_{(n)}), \\ u_{ij} &= \psi_{ij}(x, u, v, v_{(1)}, \dots, v_{(n)}), \\ &\vdots \\ u_{i_1 \dots i_n} &= \psi_{i_1 \dots i_n}(x, u, v, v_{(1)}, \dots, v_{(n)}), \end{aligned} \tag{4.5}$$

where $i, j, i_1, \dots, i_n = 1, 2$. Observe that less restrictive Bäcklund transformations are obtained if we only require

$$u_1 = \psi_1(x, u, v, v_{(1)}, \dots, v_{(n)}), \quad (4.6)$$

$$u_2 = \psi_2(x, u, v, v_{(1)}, \dots, v_{(n)}). \quad (4.7)$$

Further, observe that if (4.6), (4.7) are admitted by the differential equation

$$\omega(x, u, u_{(1)}, \dots) = 0, \quad (4.8)$$

when v satisfies the differential equation

$$\Omega(x, v, v_{(1)}, \dots) = 0, \quad (4.9)$$

then the system (4.6), (4.8), (4.9) implies (4.7) through the process of differentiation and elimination. Thus, we conclude that the less restrictive n th-order Bäcklund transformation (4.6), (4.7) admitted by (4.8), when v satisfies (4.9), is determined by (4.6).

With these observations in hand, it is now possible to establish the group nature of (4.6), (4.8) which are obtained by augmenting (4.6), (4.7) of the relations in (4.5) for classes of differential equations. In particular, for example, if we omit the x -dependence in (4.6) and take (4.8) to be of the form

$$u_1 + F(u, u_2, u_{22}, u_{222}, \dots) = 0, \quad (4.10)$$

and (4.9) to be of the form

$$v_1 + G(v, v_2, v_{22}, v_{222}, \dots) = 0, \quad (4.11)$$

then the following necessary condition is valid.

Theorem 17.2. Equation (4.10) admits the Bäcklund transformation (4.6), (4.7) if the system of equations (4.10) and (4.11) admits the operator

$$X = (u_1 - \psi) \frac{\partial}{\partial u} + \dots,$$

on the manifold of solutions of (4.8) which are invariant under the action of X , i.e. if

$$X(u_1 + F)|_{(4.6), (4.10), (4.11)} \equiv 0.$$

This result generalizes to systems of such equations and to other types of equations including those with several spatial coordinates.

In this regard one of the interesting open problems is to "invert" the above theorem. By this we mean the following: find group theoretical conditions sufficient for the existence of an equation of the form 4.7 which is compatible with (4.6) on the solutions of (4.8) and (4.9). This problem has two aspects: First, to characterize those systems (4.8), (4.9) which admit Bäcklund transformations (4.6), (4.7); Second, if the system (4.8), (4.9) admits Bäcklund transformations to find criteria under which equation (4.6) can be completed by an equation (4.7) to form a Bäcklund transformation for equations (4.8), (4.9). Observe that this problem is closely related to the problem of the linearization of a given equation, say (4.8), by means of a Bäcklund or Lie-Bäcklund transformation. In particular, a linear ansatz can be taken for the form of (4.9). Finally, we point out that the solution of this problem would complete the classical considerations of Lie, namely, given (4.6), (4.7) and (4.8) find the equation(s) such that (4.6), (4.7) is a Bäcklund transformation for equations (4.8) and (4.9).

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Paper 18

On the theory of Lie-Bäcklund transformation groups

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ABSTRACT.

The attempt to apply the theory of Lie groups to the case of Lie-Bäcklund transformations, which are certain tangent transformations of infinite order, leads to an infinite-dimensional analogue of Lie's equations. This constitutes the main difficulty in any attempt to construct an analytic theory of Lie-Bäcklund transformation groups. In this paper an algebraic solution of this difficulty by means of power series is suggested. A formal theory which preserves the principal features of Lie's theory of tangent transformations is constructed. Some applications of this theory to the group theoretic study of differential equations in which the use of Lie-Bäcklund transformations is essential are considered.

Introduction

A special place in the theory of first-order partial differential equations is occupied by Lie's tangent transformations, i.e., transformations of the form

$$x'^i = f^i(x, u, p), \quad u' = \varphi(x, u, p), \quad p'_i = \psi_i(x, u, p), \quad i = 1, \dots, n, \quad (1)$$

preserving the first-order tangency condition

$$du - p_i dx^i = 0. \quad (2)$$

This is connected with the fact that the tangent transformations act transitively on the set of first-order differential equations: an arbitrary equation $F(x, u, p) = 0$ is mapped by a tangent transformation to some first-order equation, and each solution of the original equation is taken to a solution of the new equation. Moreover, the group of all tangent transformations has no invariants, and therefore every first-order equation can be mapped to an arbitrary first-order equation of the same type by means of some tangent transformation.

The question naturally arises on possibility of constructing a similar theory of transformations for differential equations of second and higher order by considering higher-order tangent transformations or, equivalently, for systems of first-order equations by considering, instead of (1), first-order tangent transformations involving several dependent variables. This question was already posed by Sophus Lie in his fundamental paper [108] on the theory of tangent transformations and was formulated in the form of the following two questions:

1. *Are there transformations which are not tangent transformations of the form (1)-(2) and which preserve higher-order tangency conditions?*
2. *Do partial differential equations of higher order admit transformations which are not tangent transformations (1)-(2)?*

Lie predicted a negative answer to his first question and an affirmative answer to the second question. The correctness of Lie's prediction regarding the first question was confirmed by Bäcklund [12]. He showed that every transformation preserving a tangency condition of any finite order is Lie's tangent transformation. Soon afterwards Lie [110] verified that question 2 should have an affirmative answer by giving an analytic formulation of Bianchi's well-known geometric construction mapping any surface of a constant negative curvature into a surface of the same curvature. If the basic differential equation of such a surface is given in the form

$$\frac{\partial^2(2u)}{\partial x \partial y} = \sin(2u), \quad (3)$$

then the Bianchi-Lie transformation is given by the following system of first-order differential equations:

$$\frac{\partial(u + u')}{\partial x} = \sin(u - u'), \quad \frac{\partial(u - u')}{\partial y} = \sin(u + u'). \quad (4)$$

If equations (4) are compatible for a given function $u = u(x, y)$ and the function $u' = u'(x, y)$ solves the system (4), we say that the surface S defined by the equation $u = u(x, y)$ is mapped by means of (4) to the surface S' given by $u' = u'(x, y)$. As Lie has shown, this transformation is

defined if and only if S is a surface of constant curvature, i.e. if $u = u(x, y)$ satisfies (3), in which case the image u' of u under this transformation is also a solution of (3). Thus any transformation defined by (4) acts on the set of solutions of (3), leaving this equation invariant, and is not a tangent transformation.

Later Bäcklund [15] generalized Bianchi's construction and applied Lie's results to his generalization, establishing a connection with his own investigations on the consistency of over-determined systems of first-order differential equations. However, the analytic expression he obtained turned out to be a rather trivial generalization of the Bianchi-Lie transformation (4). It has the form

$$\frac{\partial(u + u')}{\partial x} = \frac{1}{a} \sin(u - u'), \quad \frac{\partial(u - u')}{\partial y} = a \sin(u + u'). \quad (5)$$

It is just the superposition of the transformation given by (4) and the one-parameter group of dilations $x \mapsto ax, y \mapsto \frac{1}{a}y$, under which equation (3) is invariant. Transformations of the form (5) are termed in the literature as *Bäcklund transformations* for equation (3). Similar transformations are also well known for a series of other nonlinear equations (see, for instance, [131]).

Bäcklund's theorem on nonexistence of tangent transformations of higher order does not exclude the possibility of existence of tangent transformations of infinite order. In fact examples of such transformations have long been known and occur frequently in the literature, for instance in connection with the Legendre transformation. In a more general context they have been studied in the papers of Bäcklund cited above. He generalizes transformations of the form (1) by considering, instead of f^i and φ , functions depending also on higher-order derivatives up to certain finite order. The transformations of derivatives are defined by consecutive differentiation of the dependent variable u' with respect to x^i . Derivatives of u' with respect to x^i of any order contain derivatives of higher order of the original variable u . Therefore, equation (2) is replaced by an infinite set of tangency conditions of all orders.

Group theoretic approach and Lie's infinitesimal method simplify the proof of Bäcklund's theorem considerably, as was shown in [68] for the case of tangent transformations of second order and in [90] and [91] in the general case. In [90] and [91] we also studied general tangent transformations of infinite order and called the *Lie-Bäcklund transformations*.

Lie-Bäcklund transformations and vector fields related to them (the *Lie-Bäcklund operators*) appear, sometimes in an implicit form, in various problems as an expression of certain symmetry properties of the problems in

question (see, e.g. [7], [83], [66], [175], [104], [162], [53]). In particular the well-known Fock transformations for the hydrogen atom (as well as their modifications in other similar problems) written in terms of the original space variables turn out to be Lie-Bäcklund transformations. The transformations (4) and (5) can also be treated as Lie-Bäcklund transformations. A detailed discussion of Bianchi-Lie, Bäcklund, and Lie-Bäcklund transformations, the relationships among them and applications to differential equations can be found in [7], where also a more detailed bibliography is given.

An attempt to extend Lie's theory to Lie-Bäcklund transformations with the goal of their infinitesimal description leads to an infinite-dimensional analogy of Lie's equations. The resulting infinite system of first-order ordinary differential equations can be reduced to a finite-dimensional system only in the cases of Lie's point and Lie tangent transformations (see [90]; one can find an interesting discussion of this question in connection with conservation laws in E. Noether's paper [133], § 3). This fact, which is actually another form of Bäcklund's theorem, leads to the considerable difficulty in constructing an analytic theory of Lie-Bäcklund transformation groups. This difficulty has been overcome in a few special cases (see [83]), but in the general it appears to be of a fundamental character.

In this paper I propose an algebraic solution of the problem. The crucial idea is to replace the analytic viewpoint in the study of continuous transformation groups by the method of formal power series. In this way we succeed in constructing a simple formal theory of groups of Lie-Bäcklund transformations, which maintains main features of Lie's theory of tangent transformations. (A similar situation is well known in the theory of infinite Lie pseudogroups; see [105].) We will also discuss applications of the theory in certain problems of the group analysis of differential equations in which the use of Lie-Bäcklund transformations is essential.

§ 1 Formal one-parameter groups

We consider the algebra of formal power series in one variable with coefficients in an infinite transcendental extension Ω of the field of real or complex numbers. An element of this algebra is said to be a *formal one-parameter group* if it has a special ("group") property. Using the isomorphism between Ω and a field of rational functions with real or complex coefficients, formal groups can be represented as power series whose coefficients are rational functions with special properties.

Let Z be the space of sequences $z = (z^i)_{i \geq 1}$ of variables z^i , $i = 1, 2, \dots$.

We consider a sequence of formal power series in one symbol a :

$$f^i(z, a) = \sum_{k=0}^{\infty} A_k^i(z) a^k, \quad i = 1, 2, \dots, \quad (1.1)$$

whose coefficients are rational functions* of a finite number of variables z^i , where different coefficients can be functions of different finite sets of these variables. We assume that

$$A_0^i(z) = z^i, \quad i = 1, 2, \dots, \quad (1.2)$$

and denote the coefficients $A_1^i(z)$ by $\xi^i(z)$,

$$\xi^i(z) = A_1^i(z). \quad (1.3)$$

One can treat (1.1) as a transformation in the space Z taking the sequence $z = (z^i)_{i \geq 1}$ into the sequence $z' = (z'^i)_{i \geq 1}$ of variables

$$z'^i = f^i(z, a) \quad i = 1, 2, \dots \quad (1.4)$$

A linear combination and the product of formal power series of the form (1.1) are defined by

$$\lambda \left(\sum_{k=0}^{\infty} A_k^i a^k \right) + \mu \left(\sum_{k=0}^{\infty} B_k^i a^k \right) = \sum_{k=0}^{\infty} (\lambda A_k^i + \mu B_k^i) a^k$$

and

$$\left(\sum_{k_1=0}^{\infty} A_{k_1}^i a^{k_1} \right) \cdot \left(\sum_{k_2=0}^{\infty} B_{k_2}^i a^{k_2} \right) = \sum_{k=0}^{\infty} \left(\sum_{k_1+k_2=k} A_{k_1}^i B_{k_2}^i \right) a^k,$$

respectively, and represent new formal power series of the form (1.1). Therefore one define the superposition of the formal transformations (1.4):

$$z''^i = f^i(z', b) = \sum_{k=0}^{\infty} A_k^i(z') b^k,$$

which, in general, takes formal power series z'^i in one symbol a into formal power series z''^i in two symbols a, b , etc. The following definition singles out a special class of formal transformations for which this increase in the number of symbols does not occur.

* *Author's note to this 2006 edition:* The theory can be extended to a more general space \mathcal{A} of *differential functions* introduced in [85] (Paper 19 in this volume), see also [86]. Consequently, $A_k^i(z)$ in (1.1) as well as invariants $F(z)$ in § 2, etc. can be differential functions, i.e. locally analytic functions of any finite number of variables z^i .

Definition 18.1. The sequence $f(z, a) = (f^i(z, a))_{i \geq 1}$ of formal power series (1.1) is called a *formal one-parameter group* if the coefficients of these power series obey the conditions

$$A_k^i(f(z, a)) = \sum_{l=0}^{\infty} \frac{(k+l)!}{k! l!} A_{k+l}^i(z) a^l, \quad i = 1, 2, \dots; \quad k = 0, 1, 2, \dots \quad (1.5)$$

Remark 18.1. The formulas (1.5) are equivalent to

$$f^i(f(z, a), b) = f^i(z, a + b), \quad i = 1, 2, \dots \quad (1.6)$$

Indeed, the left-hand sides of (1.6) are equal to

$$f^i(f(z, a), b) = \sum_{k=0}^{\infty} A_k^i(f(z, a)) b^k,$$

and the right-hand sides of (1.6) can be transformed as follows:

$$\begin{aligned} f^i(z, a + b) &= \sum_{k=0}^{\infty} A_k^i(z) (a + b)^k = \sum_{k=0}^{\infty} A_k^i(z) \sum_{k_1+k_2=k} \frac{k!}{k_1! k_2!} a^{k_1} b^{k_2} \\ &= \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{(k_1 + k_2)!}{k_1! k_2!} A_{k_1+k_2}^i(z) a^{k_1} b^{k_2} = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{(k+l)!}{k! l!} A_{k+l}^i(z) a^l \right) b^k. \end{aligned} \quad (1.7)$$

Hence the equations (1.6) have the form which is equivalent to (1.5).

If the series (1.1) converge then (1.4) defines a usual one-parameter group of transformations $T_a : z \mapsto z'$, and the group property (1.6) asserts that the superposition $T_b \circ T_a$ of the transformations corresponding to the values a and b of the group parameter is equal to the transformation T_{a+b} corresponding to the sum of these values. In the case of formal series (1.1), equation (1.6) expresses the identity of the formal series $f^i(f(z, a), b)$ in two symbols a, b and the formal series $f^i(z, a + b)$ in one symbol $c = a + b$. The substitution $b = -a$ in the formal series $f^i(f(z, a), b)$ in two symbols with the corresponding decomposition of the expression $f^i(f(z, a), -a)$ into formal power series in a leads to a formal power series in one symbol. Moreover, (1.6) yields the equations

$$f^i(f(z, a), -a) = z^i \quad i = 1, 2, \dots \quad (1.8)$$

Equations (1.8) can be regarded formally as an invertibility of formal transformations (1.4). Furthermore, one can use in the theory of formal one-parameter groups a change of the *group parameter*, $a \mapsto \bar{a}$, obtained by

replacing a in (1.1) by the formal power series

$$a = \sum_{k=1}^{\infty} \lambda_k \bar{a}^k$$

with real coefficients λ_k . In particular, the substitution $a \mapsto \lambda \bar{a}$ can be used to cancel the common constant factor in the coefficients (1.3).

Given a formal power series

$$f(z, a) = \sum_{k=0}^{\infty} A_k(z) a^k,$$

we formally set

$$A_0(z) = f(z, a)|_{a=0}.$$

Then equations (1.2) and (1.3) can be symbolically written as

$$f^i(z, a)|_{a=0} = z^i \quad i = 1, 2, \dots, \quad (1.2')$$

$$\xi^i(z) = \left. \frac{df^i(z, a)}{da} \right|_{a=0}, \quad i = 1, 2, \dots \quad (1.3')$$

We say that the sequence $\xi(z) = (\xi^i(z))_{i \geq 1}$ of rational functions in (1.3') is the *tangent vector field* of the formal one-parameter group (1.1). The following theorem extends to formal groups Lie's theorem on the connection between local one-parameter transformations groups and their tangent vector fields.

Theorem 18.1. Any sequence of formal series (1.1) representing a formal one-parameter group with a tangent vector (1.3') satisfies the differential equations

$$\frac{df^i}{da} = \xi^i(f), \quad i = 1, 2, \dots \quad (1.9)$$

Conversely, for any sequence $\xi(z) = (\xi^i(z))_{i \geq 1}$ of rational functions $\xi^i(z)$ there exists one and only one solution of (1.9) given by formal power series (1.1) and satisfying the *initial conditions* (1.2'). The solution defines a formal one-parameter group whose formal tangent vector field coincides with the given sequence $\xi(z)$.

Proof. Let a sequence of series (1.1) represent a formal one-parameter group. By definition of the derivative of a formal power series we have

$$\frac{df^i}{da} = \sum_{k=0}^{\infty} (k+1) A_{k+1}^i(z) a^k.$$

On the other hand, the property (1.5) of formal one-parameter groups yields

$$\xi^i(f) \equiv A_1^i(f(z, a)) = \sum_{k=0}^{\infty} (k+1) A_{k+1}^i A_{k+1}^i(z) a^k$$

and hence, equation (1.9) holds.

Consider now any given sequence $\xi(z)$. Plugging the series (1.1) into the differential equations (1.9), we obtain recursive formulas for the coefficients A_k^i . These recursive formulas, with initial conditions (1.2), have a unique solution. Let us show that the solution (1.1) of the problem (1.9), (1.2') satisfies the conditions (1.5). Consider the following two sequences of formal series:

$$v^i = \sum_{k=0}^{\infty} A_k^i(f(z, a)) b^k, \quad i = 1, 2, \dots, \quad (1.10)$$

$$w^i = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{(k+l)!}{k! l!} A_{k+l}^i(z) a^l \right) b^k, \quad i = 1, 2, \dots \quad (1.11)$$

Since the series $f^i(z, a)$ satisfy equations (1.9) and the initial conditions (1.2'), the following equations hold:

$$\frac{dv^i}{db} = \xi^i(v), \quad v^i|_{b=0} \equiv A_0^i(f(z, a)) = f^i(z, a), \quad i = 1, 2, \dots$$

On the other hand, according to (1.7) the series w^i can be rewritten as

$$w^i = \sum_{k=0}^{\infty} A_k^i(z) (a+b)^k, \quad i = 1, 2, \dots \quad (1.12)$$

Hence

$$\frac{dw^i}{d(a+b)} = \xi^i(w).$$

Furthermore, the representation of w^i by (1.11) yields that

$$\frac{dw^i}{db} = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{(k+l+1)!}{k! l!} A_{k+l+1}^i(z) a^l \right) b^k,$$

and their representation by (1.12), upon using transformations of the form (1.7), yields

$$\frac{dw^i}{d(a+b)} = \sum_{k=0}^{\infty} (k+1) A_{k+1}^i(z) (a+b)^k$$

$$= \sum_{k=0}^{\infty} A_{k+1}^i(z) \sum_{k_1+k_2=k} \frac{(k+1)!}{k_1! k_2!} a^{k_1} b^{k_2} = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{(k+l+1)!}{k! l!} A_{k+l+1}^i(z) a^l \right) b^k.$$

Therefore

$$\frac{dw^i}{db} = \frac{dw^i}{d(a+b)},$$

and hence the sequence of formal series (1.11) satisfies the system of differential equations

$$\frac{dw^i}{db} = \xi^i(w), \quad i = 1, 2, \dots$$

From (1.11) we also have

$$w^i|_{b=0} \equiv \sum_{l=0}^{\infty} A_l^i(z) a^l = f^i(z, a).$$

Thus, the formal power series (1.10) and (1.11) satisfy (with respect to the symbol b) the same system of differential equations of type (1.9) and the same initial conditions. Hence, by the uniqueness of the solution of the problem (1.9), ((1.2')), the series (1.10) and (1.11) coincide. It follows that the solution of (1.9), ((1.2')) satisfies (1.5). Finally, the fact that the tangent vector field of the formal one-parameter group thus obtained coincides with the original sequence $\xi(z)$ follows immediately from the recursive formulas.

§ 2 Invariants and invariant manifolds

The action of the formal transformations (1.4) can be extended to rational functions $F(z)$ in an arbitrary finite number of variables z^i by the formula $F(z) \mapsto F(z')$. Here $F(z')$ is a formal power series in a . Let

$$F(z') = \sum_{k=0}^{\infty} B_k(z) a^k.$$

The coefficients of this series are rational functions and can be expressed in terms of $F(z)$ and the coefficients of the series (1.1). In particular, in view of equations (1.2) and (1.3),

$$B_0(z) = F(z), \quad B_1(z) = \xi^i(z) \frac{\partial F(z)}{\partial z^i}.$$

Hence

$$F(z') = F(z) + a \xi^i(z) \frac{\partial F(z)}{\partial z^i} + a^2 F_1(z, a), \quad (2.1)$$

where $F_1(z, a)$ is also a formal power series of type (1.1). Using the convenient symbolic notation introduced above, we therefore obtain

$$F(z')|_{a=0} = F(z), \quad (2.2)$$

$$\frac{dF(z')}{da} \Big|_{a=0} = \xi^i(z) \frac{\partial F(z)}{\partial z^i}. \quad (2.3)$$

The function $F(z)$ is said to be an *invariant* of the formal transformations (1.4) if $F(z') = F(z)$.

Theorem 18.2. The rational function $F(z)$ is an invariant of the one-parameter group of transformations (1.4) if and only if

$$\xi^i(z) \frac{\partial F(z)}{\partial z^i} = 0. \quad (2.4)$$

Proof. The validity of condition (2.4) for the invariant $F(z)$ is insured by (2.3). Now let $F(z)$ be an arbitrary rational function satisfying (2.4). Since (2.4) is an algebraic identity, we also have

$$\xi^i(z') \frac{\partial F(z')}{\partial z'^i} = 0. \quad (2.5)$$

Furthermore, since

$$\frac{dF(z')}{da} = \frac{\partial F(z')}{\partial z'^i} \cdot \frac{df^i(z, a)}{da},$$

equations (1.9) yield

$$\frac{dF(z')}{da} = \xi^i(z') \frac{\partial F(z')}{\partial z'^i}. \quad (2.6)$$

Equations (2.5) and 2.6) show that the formal series $F(z')$ satisfies the differential equation

$$\frac{dF(z')}{da} = 0. \quad (2.7)$$

Since the solution of equation (2.7) with the initial condition (2.2) is uniquely determined, we conclude that $F(z') = F(z)$ thus proving the theorem.

Likewise one can prove the formal version of the infinitesimal test for the invariance of manifolds. Let us consider a manifold $M \subset Z$ given by a system of equations

$$\psi_\nu(z) \equiv z^{i_\nu} + \varphi_\nu(\bar{z}) = 0, \quad \nu = 1, \dots, p, \quad (2.8)$$

where $\varphi_\nu(\bar{z})$ are rational functions, $z = (z^{i_1}, \dots, z^{i_p}, \bar{z})$, the i_ν are natural numbers, and $p \leq \infty$. The manifold M is said to be *invariant* under the formal transformations (1.4) if for each solution $z = (z^i)_{i \geq 1}$ of (2.8) the image $z' = (f^i(z, a))_{i \geq 1}$ of z under the transformation also satisfies (2.8).

Theorem 18.3. The manifold M given by the system of equations (2.8) is invariant under the formal one-parameter group of transformations (1.4) if and only if the following equations hold for each $z \in M$:

$$\xi^i(z) \frac{\partial \psi_\nu(z)}{\partial z^i} = 0, \quad \nu = 1, \dots, p. \quad (2.9)$$

Proof. If the manifold M is invariant, then the equations $\psi_\nu(z) = 0$ and $\psi_\nu(z') = 0$ are satisfied for every point $z \in M$ and for each $\nu = 1, \dots, p$. Therefore the decomposition of each series $\psi_\nu(z')$ according to (2.1), namely,

$$\psi_\nu(z') = \psi_\nu(z) + a \xi^i(z) \frac{\partial \psi_\nu(z)}{\partial z^i} + a^2 \psi_{\nu 1}(z, a),$$

provides equation (2.9).

Let us assume now that manifold M satisfies (2.9). Consider an arbitrary point $z = (z^i)_{i \geq 1}$ in M and its image $z' = (f^i(z, a))_{i \geq 1}$. According to (2.6), (2.2) and (2.8), the formal series $\psi_\nu(z')$ satisfy the system of equations

$$\frac{d\psi_\nu(z')}{da} = \xi^i(z') \frac{\partial \psi_\nu(z')}{\partial z'^i}, \quad \nu = 1, \dots, p, \quad (2.10)$$

and the initial conditions

$$\psi_\nu(z')|_{a=0} = 0, \quad \nu = 1, \dots, p. \quad (2.11)$$

Since $\psi_\nu(z')$ and the right-hand sides of (2.10) are rational functions in z' , the solution of problem (2.10), (2.11) in the form of formal power series in a is defined by the recursive formulas and is unique. Note that according to (2.9) the right-hand sides of (2.10) are zero if $\psi_\nu(z') = 0$, $\nu = 1, \dots, p$. Therefore the identically vanishing series $\psi_\nu(z')$ satisfy the conditions (2.10) and (2.11). Hence, the unique solution of our problem is given by $\psi_\nu(z') = 0$, $\nu = 1, \dots, p$. Therefore the image z' of an arbitrary point $z \in M$ satisfies equations (2.8), i.e. $z' \in M$.

§ 3 The tangent structure

Let Z be a trivial bundle

$$Z = R^n \times V,$$

where the fiber $V = \sum_{s \geq 0} V^s$ is a graded vector space with finite-dimensional homogeneous components V^s , $s = 0, 1, 2, \dots$. The elements of V are defined by their homogeneous coordinates u^α , $u_{i_1 \dots i_s}^\alpha$, where u^α are coordinates of

homogeneous elements of degree 0 and $u_{i_1 \dots i_s}^\alpha$ are coordinates of homogeneous elements of degree $s \geq 1$. All indices range over positive integers: $\alpha = 1, \dots, m$; $i_1, \dots, i_s = 1, \dots, n$. The quantities $u_{i_1 \dots i_s}^\alpha$ with $s \geq 2$ are symmetric with respect to the subscripts.

The elements of the space V will also be written in the abbreviated form $u = (u^1, \dots, u^m)$, $u_{(s)} = (u_{i_1 \dots i_s}^\alpha | \alpha = 1, \dots, m; i_1, \dots, i_s = 1, \dots, n)$. Sometimes, for the sake of uniformity of notation we shall write $u_{(0)}$ for u . Elements of V will be denoted by y , where $y = (u, u_{(1)}, u_{(2)}, \dots)$. For $z \in Z$ we shall write $z = (x, y) = (x, y, u_{(1)}, u_{(2)}, \dots)$, where $x = (x^1, \dots, x^n) \in R^n$.

The linear transformation $D_i : V \rightarrow V$ ($1 \leq i \leq n$) defined by

$$D_i(u^\alpha) = u_i^\alpha, \quad D_i(u_{i_1 \dots i_s}^\alpha) = u_{ii_1 \dots i_s}^\alpha, \quad s = 1, 2, \dots,$$

satisfies the condition $D_i(V^s) \subset V^{s+1}$ for all $s \geq 0$. Hence it is an endomorphism of degree 1. The endomorphism D_i can be formally written as the differential operator

$$D_i = u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ii_1}^\alpha \frac{\partial}{\partial u_{i_1}^\alpha} + u_{ii_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots$$

acting on arbitrary rational functions in the variables $u, u_{(1)}, u_{(2)}, \dots$. In particular, one can apply it to the elements $y \in V$ since the representation $y = \sum u_{(s)}$, and hence the expression $D_i(y)$ are finite sums.

If we deal with functions depending on $z = (x, y)$, we extend the operator D_i by means of *partial differentiation* with respect to the variable x^i . The result is the operator of *total differentiation* with respect to x^i . It is also denoted by D_i and has the form

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ii_1}^\alpha \frac{\partial}{\partial u_{i_1}^\alpha} + u_{ii_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots$$

Let $T(Z)$ be the tangent bundle of the space Z . The elements of the tangent space

$$Z_z = R_x^n \times V_y$$

of Z at the point $z = (x, y)$ will be denoted by $dz = (dx, dy)$, where $dx \in R_x^n$ and $dy \in V_y$. It is convenient to identify Z_z with another copy of the space Z . Let V_y^s ($s = 0, 1, 2, \dots$) be the homogeneous components of the space V_y , where the grading of V_y is that inherited by identifying V_y with V . Elements dy in V_y will be written in the form

$$dy = (du, du_{(1)}, du_{(2)}, \dots),$$

where

$$du = (du^1, \dots, du^m) \in V_y^0,$$

$$du_{(s)} = (du_{i_1 \dots i_s}^\alpha | \alpha = 1, \dots, m; i_1, \dots, i_s = 1, \dots, n) \in V_y^s$$

are the homogeneous elements of degree 0 and $s \geq 1$, respectively.

Let us take in each spaces V_y^s ($s = 0, 1, 2, \dots$) the subspace generated by elements of the form

$$du^\alpha = u_i^\alpha dx^i, \quad du_{i_1 \dots i_s}^\alpha = u_{ii_1 \dots i_s}^\alpha dx^i,$$

where dx^i are the coordinates of the elements $dx \in R_x^n$. We thereby arrive at a subbundle of the tangent bundle $T(Z)$ which is a vector bundle over Z and has the dimension n , i.e. has n -dimensional fibers. This subbundle is called the *tangent structure* of the infinite order and is denoted by the symbol Z^T . Thus the tangent structure Z^T is given by the equations

$$\begin{aligned} du^\alpha - u_i^\alpha dx^i &= 0, \\ du_{i_1 \dots i_s}^\alpha - u_{ii_1 \dots i_s}^\alpha dx^i &= 0, \quad s = 1, 2, \dots \\ (\alpha = 1, \dots, m; i_1, \dots, i_s = 1, \dots, n). \end{aligned} \quad (3.1)$$

§ 4 The Lie-Bäcklund transformations

Let F be a vector bundle over a space B , and G a Lie group. F is said to be a *vector G -bundle over the G -space B* if

- 1) G acts continuously on the spaces F and B ,
- 2) the projection $P : F \rightarrow B$ is *equivariant* with respect to the actions of G on the spaces F and B (P commutes with the actions of G , i.e. $Pg = gP$, $g \in G$),
- 3) for an arbitrary point $b \in B$ every element $g \in G$ defines a linear transformation of the fiber F_b over b onto the fiber $F_{g(b)}$ over $g(b)$.

For a local (and therefore also formal) Lie group G the corresponding notion is defined in the obvious way.

We now consider a formal one-parameter group G of transformations

$$\begin{aligned} x'^i &= f^i(z, a), \\ u'^\alpha &= \varphi^\alpha(z, a), \\ u'_{i_1 \dots i_s}^\alpha &= \psi_{i_1 \dots i_s}^\alpha(z, a), \quad s = 1, 2, \dots, \end{aligned} \quad (4.1)$$

acting in the space $Z = R^n \times V$ of sequences $z = (x, u, u_{(1)}, u_{(2)}, \dots)$. According to § 1, each of the functions $f^i(z, a)$, $\varphi^\alpha(z, a)$ and $\psi_{i_1 \dots i_s}^\alpha(z, a)$ is a formal power series in a with coefficients in the field of rational functions in variables $x, u, u_{(1)}, u_{(2)}, \dots$. The tangent bundle $T(Z)$ is a vector G -bundle over the G -space Z if the action of G is extended to elements of the tangent space by the formulas

$$\begin{aligned}
dx'^i &= \frac{\partial f^i}{\partial x^j} dx^j + \frac{\partial f^i}{\partial u^\beta} du^\beta + \frac{\partial f^i}{\partial u_j^\beta} du_j^\beta + \dots, \\
du'^\alpha &= \frac{\partial \varphi^\alpha}{\partial x^j} dx^j + \frac{\partial \varphi^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \varphi^\alpha}{\partial u_j^\beta} du_j^\beta + \dots, \\
du_{i_1 \dots i_s}'^\alpha &= \frac{\partial \psi_{i_1 \dots i_s}^\alpha}{\partial x^j} dx^j + \frac{\partial \psi_{i_1 \dots i_s}^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \psi_{i_1 \dots i_s}^\alpha}{\partial u_j^\beta} du_j^\beta + \dots, \quad s = 1, 2, \dots
\end{aligned} \tag{4.2}$$

The right-hand sides of (4.2) contain only a finite number of nonzero summands and are formal power series in a with coefficients in the field of rational functions in variables

$$x, u, u_{(1)}, u_{(2)}, \dots, dx, du, du_{(1)}, du_{(2)}, \dots$$

Therefore the transformations (4.2) are of the same kind as (4.1).

Definition 18.2. The formal transformation (4.1) is called a *Lie-Bäcklund transformation* if the manifold defined by equations (3.1) is invariant under the transformations (4.1), (4.2).

Remark 18.2. Let G be a formal one-parameter group of Lie-Bäcklund transformations. In view of the invariance of equations (3.1) under the transformations (4.1), (4.2) the action of G can be extended to the tangent structure Z^T . Then Z^T becomes a vector G -bundle over the G -space Z . This property is a characteristic attribute of Lie-Bäcklund transformation groups and permits an obvious geometric generalization. Let F be an infinite-dimensional vector bundle, locally given as the trivial bundle Z of § 3, and let F^T be the corresponding tangent structure, i.e. the n -dimensional subbundle of the tangent bundle $T(F)$ locally given by the procedure described in § 3. The Lie group G acting on F is called the group of Lie-Bäcklund transformations if F^T is a vector G -bundle over the G -space F with respect to the action of the group G on F^T induced by the action of G on the tangent bundle $T(F)$.

Consider a formal one-parameter group G of transformations (4.1). The tangent vector field of this group will be written in the form

$$\Xi = (\xi^i, \eta^\alpha, \zeta_{i_1}^\alpha, \zeta_{i_1 i_2}^\alpha, \dots), \tag{4.3}$$

where, according to (1.3'),

$$\xi^i = \left. \frac{df^i}{da} \right|_{a=0}, \quad \eta^\alpha = \left. \frac{d\varphi^\alpha}{da} \right|_{a=0}, \quad \zeta_{i_1 \dots i_s}^\alpha = \left. \frac{d\psi_{i_1 \dots i_s}^\alpha}{da} \right|_{a=0}, \quad s = 1, 2, \dots \tag{4.4}$$

We now examine the action (4.1), (4.2) of the group G on the tangent bundle $T(Z)$. The tangent vector field of the *extended* action (4.1), (4.2) of G can be written in the form

$$\tilde{\Xi} = (\Xi, \tilde{\xi}^i, \tilde{\eta}^\alpha, \tilde{\zeta}_{i_1}^\alpha, \tilde{\zeta}_{i_1 i_2}^\alpha, \dots). \quad (4.5)$$

Its coordinates $\tilde{\xi}^i, \tilde{\eta}^\alpha, \tilde{\zeta}_{i_1}^\alpha, \dots$ appearing in addition to the coordinates of (4.3), are defined by

$$\tilde{\xi}^i = \frac{d(dx'^i)}{da} \Big|_{a=0}, \quad \tilde{\eta}^\alpha = \frac{d(du'^\alpha)}{da} \Big|_{a=0}, \quad \tilde{\zeta}_{i_1 \dots i_s}^\alpha = \frac{d(du_{i_1 \dots i_s}'^\alpha)}{da} \Big|_{a=0}.$$

In view of (4.2) and (4.4) they are equal to

$$\begin{aligned} \tilde{\xi}^i &= \frac{\partial \xi^i}{\partial x^j} dx^j + \frac{\partial \xi^i}{\partial u^\beta} du^\beta + \frac{\partial \xi^i}{\partial u_j^\beta} du_j^\beta + \dots, \\ \tilde{\eta}^\alpha &= \frac{\partial \eta^\alpha}{\partial x^j} dx^j + \frac{\partial \eta^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \eta^\alpha}{\partial u_j^\beta} du_j^\beta + \dots, \\ \tilde{\zeta}_{i_1 \dots i_s}^\alpha &= \frac{\partial \zeta_{i_1 \dots i_s}^\alpha}{\partial x^j} dx^j + \frac{\partial \zeta_{i_1 \dots i_s}^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \zeta_{i_1 \dots i_s}^\alpha}{\partial u_j^\beta} du_j^\beta + \dots \end{aligned} \quad (4.6)$$

To each vector field (4.3) there corresponds, according to Theorem 18.1, a formal one-parameter group of transformations of the form (4.1). The following theorem (see [90] and [91]) identifies those vector fields (4.3) to which correspond groups of Lie-Bäcklund transformations, and thereby gives an infinitesimal characterization of Lie-Bäcklund transformations.

Theorem 18.4. A formal one-parameter group G of transformations (4.1) is a group of Lie-Bäcklund transformations if and only if the tangent vector field (4.3) of G satisfies the conditions

$$\zeta_{i_1 \dots i_s}^\alpha = D_{i_1} \dots D_{i_s} (\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s = 1, 2, \dots \quad (4.7)$$

Proof. According to Theorem 18.3, the manifold (3.1) is invariant under the transformations (4.1), (4.2) if and only if the equations

$$\begin{aligned} \tilde{\eta}^\alpha - u_i^\alpha \tilde{\xi}^i - \zeta_i^\alpha dx^i &= 0, \\ \tilde{\zeta}_{i_1 \dots i_s}^\alpha - u_{i i_1 \dots i_s}^\alpha \tilde{\xi}^i - \zeta_{i i_1 \dots i_s}^\alpha dx^i &= 0, \quad s = 1, 2, \dots, \end{aligned} \quad (4.8)$$

hold for all points $(z, dz) \in Z^T$. We derive (4.7) from (4.8) by substitution of (4.6) in (4.8) with the subsequent replacement of du^α and $du_{i_1 \dots i_s}^\alpha$ by $u_i^\alpha dx^i$ and $u_{i i_1 \dots i_s}^\alpha dx^i$, respectively, in accordance with (3.1).

According to Theorem 18.4, the vector field (4.3) of a group of Lie-Bäcklund transformations is completely determined by its coordinates $\xi = (\xi^1, \dots, \xi^n)$ and $\eta = (\eta^1, \dots, \eta^m)$. Furthermore, the coordinates ξ and η can be arbitrary rational functions. The remaining coordinates $\zeta_{i_1}^\alpha, \zeta_{i_1 i_2}^\alpha, \dots$ of the vector field (4.3) are defined by (4.7). Keeping this fact in mind and adopting the standard way of treating vector fields as differential, or *infinitesimal*, operators

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_s}^\alpha},$$

we can now write the tangent vector fields of groups of Lie-Bäcklund transformations in the abbreviated form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots \quad (4.9)$$

and call them *Lie-Bäcklund operators*.

The operator

$$X_* = \xi_*^i D_i \equiv \xi_*^i \frac{\partial}{\partial x^i} + \xi_*^i u_i^\alpha \frac{\partial}{\partial u^\alpha} + \dots, \quad (4.10)$$

where $\xi_*^i = \xi_*^i(z)$ ($i = 1, \dots, n$) are arbitrary rational functions, is a Lie-Bäcklund operator. Indeed, the expressions $\eta^\alpha - \xi^j u_j^\alpha$ for the operator (4.10) are identically zero, and it is evident that the conditions (4.7) are satisfied. The commutator

$$[X_*, X] = X_* X - X X_*$$

of the vector fields (4.9) and (4.10) equals

$$[X_*, X] = (X_*(\xi^i) - X(\xi_*^i)) D_i, \quad (4.11)$$

so that the operators of the form (4.10) generate an ideal in the Lie algebra of Lie-Bäcklund operators (4.9). Therefore, considering the corresponding quotient algebra, we can write the Lie-Bäcklund operators in the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots \quad (4.12)$$

Then the *prolongation formulas* (4.7) become

$$\zeta_{i_1 \dots i_s}^\alpha = D_{i_1} \dots D_{i_s}(\eta^\alpha). \quad (4.13)$$

Thus, there exists an equivalence relation on the set of Lie-Bäcklund operators:

$$X \sim X' \quad \text{if} \quad X - X' = X_*,$$

where X_* has the form (4.10) with some coefficients $\xi_*^i(z)$. In particular it follows from (4.11) that the operator (4.12), assuming conditions (4.13), commutes with the total differentiations D_i . Conversely, commutativity of the vector field (4.3) (with $\xi^i = 0$) with the differentiations D_i yields (4.13) (see [54]), and hence can be used for definition of Lie-Bäcklund transformations.

The transition from (4.9) to the equivalent Lie-Bäcklund operators of the form (4.12) is convenient in many problems. In some cases it can lead, however, to a loss of geometric clearness. For example the infinitesimal operator

$$X = \frac{\partial}{\partial x^i}$$

of a point transformations group, namely, the group of translations $\bar{x}^i = x^i + a$ in the variable x^i , after transition to the equivalent form (4.12) becomes

$$X' = u_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \dots$$

The formal transformations corresponding to the latter operator will be presented in Example 18.1 at the end of this section. It is desirable, therefore, to distinguish among all Lie-Bäcklund operators of the form (4.9) those to which there correspond groups of point transformations or, in the case of one *dependent* variable u (i.e. $m = 1$), Lie tangent transformations.

As regards point transformations, this problem is easy to solve: the vector fields (4.9) which are equivalent to infinitesimal operators of groups of point transformations are characterized by the relations

$$\xi^i = \xi^i(x, u) + \xi_*^i(z), \quad \eta^\alpha = \eta^\alpha(x, u) + [\xi^i(x, u) + \xi_*^i(z)]u_i^\alpha \quad (4.14)$$

with arbitrary functions $\xi^i(x, u)$, $\xi_*^i(z)$, $\eta^\alpha(x, u)$, and $\xi_*^\alpha(z)$.

Tangent transformations (1) generate a broader class of equivalent Lie-Bäcklund operators. It is characterized by the following theorem.

Theorem 18.5. When $m = 1$ the Lie-Bäcklund operators of the form (4.9) are equivalent to infinitesimal operators of one-parameter groups of Lie tangent transformations if and only if

$$\xi^i = \xi^i(x, u, u_{(1)}) + \xi_*^i(z), \quad \eta = \eta(x, u, u_{(1)}) + \xi_*^i(z)u_i. \quad (4.15)$$

Proof. The infinitesimal operators of groups of Lie tangent transformations have the form (see [108])

$$Y = -\frac{\partial W}{\partial u_i} \frac{\partial}{\partial x^i} + \left(W - u_i \frac{\partial W}{\partial u_i} \right) \frac{\partial}{\partial u} + \left(\frac{\partial W}{\partial x^i} + u_i \frac{\partial W}{\partial u} \right) \frac{\partial}{\partial u_i}, \quad (4.16)$$

where $W = W(x, u, u_{(1)})$ can be an arbitrary function. Therefore it is sufficient to show that an arbitrary Lie-Bäcklund operator of the form

$$X = \xi^i(x, u, u_{(1)}) \frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}) \frac{\partial}{\partial u} + \cdots \quad (4.17)$$

is equivalent to an operator (4.16) with some function W in variables $x, u, u_{(1)}$. For such a function one can take $W = \eta - \xi^i u_i$, where ξ^i and η are the coefficients of the operator (4.17) under consideration. Indeed, one can easily verify that

$$X - Y = \left(\xi^i + \frac{\partial W}{\partial u_i} \right) D_i$$

for the indicated choice of W .

Let us consider now several examples taking for simplicity $n = m = 1$. In this case we will write u_s for $u_{(s)}$.

Example 18.1. 1. Let the Lie-Bäcklund operator (4.12) have the coefficient $\eta = u_1$. For the construction of the corresponding formal transformations (4.1) it is necessary to solve equations (1.9). In our case, these equations, invoking (4.13), are written:

$$\frac{du'}{da} = u'_1, \quad \frac{du'_s}{da} = u'_{s+1}, \quad s = 1, 2, \dots$$

The initial conditions (1.2') have the form

$$u'|_{a=0} = u, \quad u'_s|_{a=0} = u_s, \quad s = 1, 2, \dots$$

In order to solve the problem it suffices to find a formal power series

$$u' = \sum_{k=0}^{\infty} A_k(u, u_1, u_2, \dots) a^k, \quad A_0 = u,$$

satisfying the equation

$$\frac{du'}{da} = D(u'), \quad (4.18)$$

where, in accordance with § 3, the differentiation D is written

$$D = \sum_{s=0}^{\infty} u_{s+1} \frac{\partial}{\partial u_s}.$$

The variable x is not subjected to any transformation ($x' = x$), and the transformed values u'_1, u'_2, \dots of u_1, u_2, \dots are obtained, according to the

definition of Lie-Bäcklund transformations, by differentiation of the series u' , i.e.

$$u'_1 = D(u'), \quad u'_2 = D^2(u'), \dots$$

Substitution of the series u' in (4.18) yields the recursive formula

$$(k+1)A_{k+1} = D(A_k),$$

whence, using the initial condition $A_0 = u$, one can determine uniquely the coefficients of the desired series u' . Thus the operator

$$X = u_1 \frac{\partial}{\partial u} + \dots,$$

which, as we remarked above, is equivalent to the infinitesimal operator of the group of translations $\bar{x} = x + a$, generates the following formal one-parameter group of Lie-Bäcklund transformations:

$$u' = \sum_{k=0}^{\infty} \frac{u_k}{k!} a^k, \quad u'_s = \sum_{k=0}^{\infty} \frac{u_{k+s}}{k!} a^k, \quad s = 1, 2, \dots \quad (4.19)$$

Example 18.2. As the second example we take the Lie-Bäcklund operator

$$X = u_p \frac{\partial}{\partial u} + \dots,$$

where p is an arbitrary natural number. Now, instead of (4.18), we have the equation

$$\frac{du'}{da} = D^p(u').$$

Substitution of $u' = \sum_{k \geq 0} A_k a^k$ in this equation provides the following recursive formula

$$(k+1)A_{k+1} = D^p(A_k), \quad A_0 = u.$$

Hence the Lie-Bäcklund group is given by the formal transformations

$$u' = \sum_{k=0}^{\infty} \frac{u_{pk}}{k!} a^k, \quad u'_s = \sum_{k=0}^{\infty} \frac{u_{pk+s}}{k!} a^k, \quad s = 1, 2, \dots \quad (4.20)$$

It is not difficult to clarify when (4.20) defines a usual one-parameter group of transformations, i.e. the corresponding series converge. If the series u' in (4.20) converges in a disk $|a| \leq r$, then the sequence (u, u_1, u_2, \dots) satisfies the inequalities

$$|u_{pk}| \leq C k! r^{-k}, \quad k = 0, 1, 2, \dots; \quad C = \text{const.} \quad (4.21)$$

The conditions (4.21) imply that it is possible to apply the transformations (4.20) repeatedly since the sequence (u', u'_1, u'_2, \dots) obtained after the transformation again satisfies inequalities of the type (4.21). Indeed, for $|a| < r$, using (4.20) and (4.21), we obtain

$$\begin{aligned} |u'_{ps}| &\leq \sum_{k=0}^{\infty} \frac{|a|^k}{k!} |u_{p(k+s)}| \leq C \sum_{k=0}^{\infty} \frac{(k+s)!}{k!} \left(\frac{|a|}{r}\right)^k r^{-s} = Cr^{-s} \sum_{k=0}^{\infty} \frac{(k+s)!}{k!} b^k \\ &= Cr^{-s} \left(\frac{\partial}{\partial b}\right)^s \sum_{k=0}^{\infty} b^{k+s} = Cr^{-s} \left(\frac{\partial}{\partial b}\right)^s \sum_{k=0}^{\infty} b^k = Cr^{-s} \left(\frac{\partial}{\partial b}\right)^s \frac{1}{1-b} \\ &= Cr^{-s} \frac{s!}{(1-b)^{s+1}} = C \frac{r}{r-|a|} s! (r-|a|)^{-s} \leq C_1 s! r_1^{-s}, \quad s = 0, 1, 2, \dots \end{aligned}$$

If we apply Lie-Bäcklund transformations to functions $u = u(x)$ with $u_1 = \frac{du(x)}{dx}$, $u_2 = \frac{d^2u(x)}{dx^2}$, \dots , the above inequalities show that the transformations (4.20) act as an ordinary one-parameter group of transformations in the class of entire functions defined by the conditions (4.21). The indicated class of functions coincides with the class of all analytic functions in the case $p = 1$ in agreement with the possibility of interpreting the formal transformations (4.19) as point transformations.

Example 18.3. Consider the operator $X = u_1^2 \frac{\partial}{\partial u} + \dots$. The corresponding formal group of Lie-Bäcklund transformations is given by the recursive formula $(k+1)A_{k+1} = \sum_{i+j=k} D(A_i) \cdot D(A_j)$, $A_0 = u$. In accordance with Theorem 18.5 the Lie-Bäcklund operator under consideration is equivalent to the infinitesimal operator $Y = -2u_1 \frac{\partial}{\partial x} - u_1^2 \frac{\partial}{\partial u}$ of the one-parameter group of tangent transformations

$$x' = x - 2u_1 a, \quad u' = u - u_1^2 a, \quad u'_1 = u_1.$$

§ 5 A few applications

It is a well-known phenomenon in group analysis of differential equations (see [144]) that the standard reduction of a differential equation of higher order to an equivalent system of first-order differential equations can substantially change the group properties of the original equation. The reason for this effect is obviously based on the fact that the transformation of the original equation into a first-order system is not a point transformation, while one requires that the group admitted by the original equation should be represented by point transformations. However, this explanation is incomplete. It remains unclear how new symmetries not inherent to the original equation

arise and in what these symmetries pass to under the inverse transformation. A natural answer to these questions is supplied if we consider, instead of point transformations, a wider class of Lie-Bäcklund transformations. This is due to the fact that the transformation of higher-order differential equations to equivalent systems of first-order equations belongs precisely to the class of Lie-Bäcklund transformations. In consequence, some point transformations admitted by the equations in question may become, after the transformation into the equivalent system, Lie-Bäcklund transformations admitted by the transformed systems, and vice versa. This explains the illusion of appearance or disappearance of some types of symmetries which occurs when one restricts oneself to groups of point transformations. The following simple example illustrates this phenomenon.

The second-order equation

$$\frac{d^2u}{dx^2} = 0 \quad (5.1)$$

is invariant under an 8-parameter group which is the maximal group of point transformations for this equation. The equivalent system of first-order equations

$$\frac{du^1}{dx} = u^2, \quad \frac{du^2}{dx} = 0 \quad (5.2)$$

admits an infinite group of point transformations generated by the operators

$$X_* = \xi \frac{\partial}{\partial x} + \xi u^2 \frac{\partial}{\partial u^1}, \quad (5.3)$$

$$X = (u^1 g + h) \frac{\partial}{\partial u^1} + u^2 g \frac{\partial}{\partial u^2}, \quad (5.4)$$

where $\xi = \xi(x, u^1, u^2)$, $g = g(u^1 - xu^2, u^2)$ and $h = h(u^1 - xu^2, u^2)$ are arbitrary functions of their arguments. The transformation $u^1 \mapsto u$, $u^2 \mapsto u_{(1)} \equiv \frac{du}{dx}$ connecting the system (4.2) with equation (5.1), converts the operator (5.3), extended to the derivatives u^1 and u^2 , into the trivial Lie-Bäcklund operator of type (4.10). The resulting operator is obviously admitted by equation (5.1) since every operator of the form (4.10) is admitted by any differential equation. Furthermore, the above transformation converts (5.4) into the Lie-Bäcklund operator

$$X = [ug(u - xu_{(1)}, u_{(1)}) + h(u - xu_{(1)}, u_{(1)})] \frac{\partial}{\partial u} + \dots, \quad (5.5)$$

which, as is easily verified, also is admitted by equation (5.1). Among the operators of the form (5.5) there are, in particular, all infinitesimal operators of point transformations admitted by equation (5.1). Furthermore, one

can show that the operators of the form (5.5) exhaust the class of nontrivial Lie-Bäcklund operators admitted by (5.1). On the other hand, although not every operator of type (5.5) is the infinitesimal operator of a group of Lie tangent transformations, every operator (5.5) is in fact, according to Theorem 18.5, equivalent to an infinitesimal operator of Lie tangent transformations. Therefore we can conclude that the appearance of additional symmetries resulting from replacing (5.1) by the equivalent system of first-order differential equations (5.2) is connected with existence of an infinite group of Lie tangent transformations admitted by equation (5.1).

The invariance of the group property of differential equations under equivalence transformations opens new possibilities for using group theoretic considerations in the study of linearization of partial differential equations. In the case of equations which can be linearized by an ordinary change of variables (i.e. by means of point transformation) the situation is clear; this is only possible in the case when the equation under consideration admits an infinite group of point transformations where the *degree of infinity* of the group is defined by the *size of the set of solutions* of the original equation. However, many methods of linearization occur in practice which are not given by point transformations and which therefore cannot be treated a priori by group theoretic methods when one restricts oneself to groups of point transformations. One can take, for example, the famous Hopf transformation

$$u = -2 \frac{v_x}{v}. \quad (5.6)$$

It linearizes Burger's equation

$$u_t + uu_x - u_{xx} = 0, \quad (5.7)$$

taking it into the heat equation

$$u_t - v_{xx} = 0. \quad (5.8)$$

The group properties of these equations are well known. Equation (5.7) is invariant under the 5-parameter group of point transformations specific in one-dimensional gas dynamics for the special gas with the polytropic index $\gamma = 3$, whereas equation (5.8), being linear and homogeneous, admits dilations of v and the infinite group of point transformations generated by the operator

$$X = h(t, x) \frac{\partial}{\partial v}, \quad (5.9)$$

where $h(t, x)$ is an arbitrary solution of (5.8). Furthermore, the transformation (5.6) takes the above-mentioned 5-parameter group for Burger's

equation into a group admitted by (5.8) preserving its pointwise character. This difference in the group properties makes obvious the impossibility of linearizing (5.7) by point transformations and, in particular, of comparing these two equations without the use of nonpoint transformations. The use of Lie-Bäcklund transformations allows one to extend group theoretic considerations to certain types of linearization by means of non-point transformations. The transformation (5.6) maps the operator (5.9) into the Lie-Bäcklund operator

$$X = (hu + 2h_x)e^{\varphi/2} \frac{\partial}{\partial u} + \cdots, \quad \varphi_x = u \quad (5.10)$$

admitted by equation (5.7) for an arbitrary function $h(t, x)$ satisfying (5.8). The existence of such an infinite group of Lie-Bäcklund transformations serves as the group theoretic foundation of the linearization of Burger's equation. However, it should be pointed out that, because of its dependence on the potential φ , the operator (5.10) has a nonlocal character not envisioned above. But it is possible to avoid this difficulty by noting that $(h\varphi_x + 2h_x)e^{\varphi/2} = 2(h e^{\varphi/2})_x$ and rewriting (5.7) in terms of the potential. Substituting $u = \varphi_x$ in (5.7) and integrating it once, we obtain, taking into account non-uniqueness of the choice of φ , the equivalent equation

$$\varphi_t + \frac{1}{2}\varphi_x^2 - \varphi_{xx} = 0. \quad (5.11)$$

Then we obtain, instead of (5.10), the operator $X = h(t, x)e^{\varphi/2} \frac{\partial}{\partial \varphi}$ generating an infinite group of point transformations admitted by (5.11). Therefore, it is clear from the group theoretic viewpoint that equation (5.11) is linearizable. Since the Hopf transformation is now the point transformation

$$v = e^{-\varphi/2}, \quad \varphi = -2 \ln v \quad (5.12)$$

connecting the equations (5.8) and (5.11), these two equations admit similar groups of point transformations.

Another widely known example is provided by the equation for stationary transonic motion of gas:

$$\varphi_x \varphi_{xx} + \varphi_{yy} = 0. \quad (5.13)$$

It can be linearized by the hodograph method. A discussion of the group theoretic aspects of this method can be found in [138] and [24]. In this connection we should point out that it is not possible to decide on the basis of the invariance properties of equation (5.13) under the group of point transformations whether or not it is possible to linearize this equation,

since the latter only admits a 6-parameter group. However a possibility of linearization can be easily seen on the basis of invariance properties of (5.13) under a group of Lie-Bäcklund transformations. Taking a Lie-Bäcklund operator (4.12) of the special type

$$X = \Phi(\varphi_x, \varphi_y) \frac{\partial}{\partial \varphi} + \cdots \quad (5.14)$$

one obtains, as condition for the invariance of equation (5.13) under the formal one-parameter group with tangent vector (5.14), the second-order linear equation

$$\Phi_{uu} + u\Phi_{vv} = 0, \quad (5.15)$$

where $u = \varphi_x$, $v = \varphi_y$. The existence of this infinite group of Lie-Bäcklund transformations not only hints the possibility of linearizing (5.13) but also realizes this possibility by means of (5.15). Since Theorem 18.5 is applicable to the operator (5.14), one can also realize the linearization by means of tangent transformations, which is well known in mechanics. After we carry out the equivalence transformation indicated in Theorem 18.5 the operator (5.14) takes the form of the infinitesimal operator

$$X = \Phi_u \frac{\partial}{\partial x} + \Phi_v \frac{\partial}{\partial y} + (u\Phi_u + v\Phi_v - \Phi) \frac{\partial}{\partial \varphi}$$

of the following group of tangent transformations:

$$\begin{aligned} x' &= x + a\Phi_u, & y' &= y + a\Phi_v, \\ \varphi' &= \varphi + a(u\Phi_u + v\Phi_v - \Phi), \\ u' &= u, & v' &= v; \end{aligned}$$

where, as above, we employ the abbreviations $u = \varphi_x$ and $v = \varphi_y$.

Let us make a few remarks on a question connected with the use of Lie-Bäcklund transformations in the search for conservation laws. Let a given system of differential equations S admit a group G of Lie-Bäcklund transformations. As in the case of groups of point transformations, every transformation of G takes an arbitrary solution of S into another solution. Hence we have an action of G on the set of conservation laws of the differential equations being considered. Therefore, starting with a certain conservation law one can obtain new conservation laws by applying the Lie-Bäcklund transformations to the given conservation law. New conservation laws, generally speaking, can form together with the original conservation law a linearly independent system. It is therefore natural to pose the problem of finding a basis (specifically, a G -basis) of conservation laws, i.e. a

minimal set of conservation laws from which one can obtain all others by means of transformations of the group G and linear combinations. It would be desirable in this context to find out whether or not there exists a finite basis of conservation laws for each system of differential equations when we choose G to be the group of all Lie-Bäcklund transformations admitted by the system S , or a sufficiently large subgroup of G .

The method of constructing a basis of conservation laws is illustrated here by means of the equations of gas dynamics,

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p &= 0, \\ \rho_t + \mathbf{v} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{v} &= 0, \\ p_t + \mathbf{v} \cdot \nabla p + \gamma p \operatorname{div} \mathbf{v} &= 0, \end{aligned} \quad (5.16)$$

with the special value of the polytropic index $\gamma = (n + 2)/n$, where $n \leq 3$ is the number of coordinates of the spatial variable $\mathbf{x} = (x^1, \dots, x^n)$. It is known (see [138]) that in this case the group admitted by the system (5.16) is larger compare to the group admitted by gasdynamic equations with an arbitrary γ . Namely, the group is extended the one-parameter group with the infinitesimal operator

$$X = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} + (x^i - tv^i) \frac{\partial}{\partial v^i} - (n + 2)tp \frac{\partial}{\partial p} - nt\rho \frac{\partial}{\partial \rho}. \quad (5.17)$$

The result of this extension of the group of symmetries is a larger number of conservation laws in gas dynamics. This was shown in [80] by means of E. Noether's theorem, and in [171] by a direct construction of conservation laws. Namely, the laws of conservation of mass m , energy E , momentum \mathbf{P} , angular momentum \mathbf{M} and the law of the motion of the center of mass \mathbf{C} given by

$$\frac{d}{dt} \int_{\Omega(t)} \rho d\omega = 0, \quad (\mathbf{m})$$

$$\frac{d}{dt} \int_{\Omega(t)} \left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{p}{\gamma - 1} \right) d\omega = - \int_{S(t)} p \mathbf{v} \cdot \boldsymbol{\nu} dS, \quad (\mathbf{E})$$

$$\frac{d}{dt} \int_{\Omega(t)} \rho \mathbf{v} d\omega = - \int_{S(t)} p \boldsymbol{\nu} dS, \quad (\mathbf{P})$$

$$\frac{d}{dt} \int_{\Omega(t)} \rho (\mathbf{x} \times \boldsymbol{\nu}) d\omega = - \int_{S(t)} p (\mathbf{x} \times \boldsymbol{\nu}) dS, \quad (\mathbf{M})$$

and

$$\frac{d}{dt} \int_{\Omega(t)} \rho(t\mathbf{v} - \mathbf{x}) d\omega = - \int_{S(t)} tp \boldsymbol{\nu} dS, \quad (\mathbf{C})$$

respectively, hold in the case of arbitrary γ . If $\gamma = (n+2)/n$, there exist the following two conservation laws specific to this for the case $\gamma = (n+2)/n$:

$$\frac{d}{dt} \int_{\Omega(t)} [t(\rho|\mathbf{v}|^2 + np) - \rho\mathbf{x} \cdot \mathbf{v}] d\omega = - \int_{S(t)} p(2t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS, \quad (5.18)$$

$$\frac{d}{dt} \int_{\Omega(t)} [t^2(\rho|\mathbf{v}|^2 + np) - \rho\mathbf{x} \cdot (2t\mathbf{v} - \mathbf{x})] d\omega = - \int_{S(t)} 2tp(t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS, \quad (5.19)$$

We employ here the integral form of conservation laws commonly used in fluid dynamics and the following notation: $\Omega(t)$ is an arbitrary n -dimensional domain, moving together with the gas, $S(t)$ is the boundary of $\Omega(t)$ and $\boldsymbol{\nu}$ is the outer unit normal to $S(t)$.

We now provide a basis of the conservation laws (refotm - (5.19) with respect to the well-known (see [138]) group G of point transformations admitted by equations (5.16). In the calculations it is convenient to represent the group G by Lie-Bäcklund transformations rewriting the corresponding infinitesimal operators of the group G in the equivalent form (4.12). Then, evidently, a conservation law for (5.16) will go over under the action of the corresponding operator of type (4.12) into a conservation law. In what follows, for convenience we will write the expressions for the conserved densities (i.e. the integrands on the left-hand sides of the conservation laws given above) up to inessential divergent summands. To illustrate the characteristic features of these computations let us consider the operator (5.17). The equivalent form (4.12) of this operator is

$$Y = (t^2 v_t^i + t\mathbf{x} \cdot \nabla v^i + tv^i - x^i) \frac{\partial}{\partial v^i} + t(tp_t + \mathbf{x} \cdot \nabla p + (n+2)p) \frac{\partial}{\partial p} + t(t\rho_t + \mathbf{x} \cdot \nabla \rho + n\rho) \frac{\partial}{\partial \rho}. \quad (5.20)$$

Under the action of the operator (5.20) the law of conservation of energy becomes the special conservation law (5.18). Indeed, using (5.16) we obtain

$$\begin{aligned} Y(\rho|\mathbf{v}|^2 + np) &= |\mathbf{v}|^2 [t^2 \rho_t + \operatorname{div}(t\rho\mathbf{x})] \\ &\quad + n[t^2 p_t + 2tp + \operatorname{div}(tp\mathbf{x})] + 2\rho\mathbf{v} \cdot [t^2 \mathbf{v}_t + t(\mathbf{x} \cdot \nabla)\mathbf{v} + t\mathbf{v} - \mathbf{x}] \\ &= 2[t(\rho|\mathbf{v}|^2 + np) - \rho\mathbf{x} \cdot \mathbf{v}] + \operatorname{div}[t(\rho|\mathbf{v}|^2 + np)(\mathbf{x} - t\mathbf{v}) - 2t^2 p\mathbf{v}]. \end{aligned}$$

Likewise one can show that the conservation law (5.19) is obtained from (5.18) by the action of the operator (5.20). Moreover, using the Galilean transformations one can perform the following transformations of conservation laws:

$$E \longmapsto \mathbf{P}, \quad \mathbf{P} \longmapsto m, \quad \mathbf{M} \longmapsto \mathbf{C}.$$

Furthermore, by using rotations, one can map any one of the components of the angular momentum \mathbf{M} (see equation (5.20)), e.g. M_3 , to its other components. The two remaining conserved quantities, namely the energy E and the component M_3 of the angular momentum, cannot be transformed into one another by the action of G . Thus we arrive at the following statement.

The conservation of energy E and M_3 provide a basis of conservation laws for the equations of gas dynamics.

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Paper 19

Sur l'équivalence des équations d'évolution, qui admettent une algèbre de Lie-Bäcklund infinie

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Le but de cette Note est de montrer comment on peut construire des transformations d'équivalence (réalisées par des transformations de Lie-Bäcklund [91]) pour des équations d'évolution, qui admettent une algèbre infinie. En particulier, on établit que les équations $u_t = u^3 u_{xxx}$ (ou $z_t = (z^{-1/2})_{xxx}$, $z = u^{-2}$) et $v_t = v_{yyy} + vv_y$, considérées usuellement comme des équations “intégrables” différentes, sont équivalentes.

The aim of this Note is to show how one can construct equivalence transformations (using Lie-Bäcklund transformations [91]) for evolution equations, which allow an infinite algebra. In particular it is established that the equations $u_t = u^3 u_{xxx}$ (or $z_t = (z^{-1/2})_{xxx}$, $z = u^{-2}$) and $v_t = v_{yyy} + vv_y$, usually considered as different “integrable” equations, are equivalent.

§ 1 Algèbres \mathcal{A}_F et \mathcal{L}_F

Soit x une variable indépendante, D une dérivation, u une indéterminée différentielle [154] avec des dérivées successives u_1, u_2, \dots , c'est-à-dire des indéterminées telles que $D(u_i) = u_{i+1}$, $u_0 = u$. Une fonction analytique $f = f(x, u, u_1, \dots, u_n)$, $n < \infty$, $\frac{\partial f}{\partial u_n} \neq 0$, est dite une fonction différentielle

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d'ordre n . On considère l'espace \mathcal{A} des fonctions différentielles d'ordre fini muni de la dérivation

$$D \equiv D_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i}$$

et de deux structures algébriques, une structure d'algèbre associative donnée par le produit usuel des fonctions et une structure d'algèbre de Lie donnée par le crochet:

$$\{f, g\} = f_*g - g_*f, \quad \text{où} \quad f_* = \sum_{i \geq 0} f_i D^i, \quad f_i = \frac{\partial f}{\partial u_i}.$$

Des éléments de \mathcal{A} peuvent dépendre d'un paramètre t .

Une équation d'évolution:

$$u_t = F(x, u, u_1, \dots, u_m), \quad m \geq 2 \quad (1.1)$$

est considérée comme un système dynamique:

$$\frac{du_i}{dt} = D^i(F), \quad i = 0, 1, 2, \dots, \quad (1.2)$$

dans \mathcal{A} . On désigne par:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i \geq 0} D^i(F) \frac{\partial}{\partial u_i}, \quad (1.3)$$

la dérivation le long des trajectoires de (1.2). L'algèbre de Lie-Bäcklund pour l'équation (1.1) est une sous-algèbre \mathcal{A}_F de l'algèbre de Lie \mathcal{A} définie par:

$$\mathcal{A}_F = \left\{ f \in \mathcal{A} : \frac{\partial f}{\partial t} - \{F, f\} = 0 \right\}.$$

L'algèbre \mathcal{A}_F donne naissance par le système dynamique:

$$\frac{du_i}{d\tau} = D^i(F), \quad f \in \mathcal{A}_F, \quad i = 0, 1, 2, \dots,$$

au groupe formel (en paramètre τ) des transformations de Lie-Bäcklund [84], qui laisse (1.1) invariante. On ne considère ici que des équations, qui admettent une algèbre de Lie-Bäcklund infinie telle qu'elle contient des éléments d'ordre arbitrairement grand.

Soit \mathcal{L} une algèbre associative des opérateurs formels:

$$L = \sum_{i=-\infty}^n a_i D^i, \quad a_i \in \mathcal{A}; \quad (1.4)$$

le nombre entier n est dit l'ordre de l'opérateur (1.4). La multiplication dans \mathcal{L} est définie par la règle de Leibniz et la formule

$$D^{-1}a = aD^{-1} - D(a)D^{-2} + D^2(a)D^{-3} - \dots$$

A (1.1) correspond une sous-algèbre $\mathcal{L}_F \subset \mathcal{L}$ définie par l'équation:

$$L_t = F_*L - LF_*, \quad (1.5)$$

où L_t est obtenu par la différentiation (1.3) des coefficients de l'opérateur L . L'algèbre \mathcal{L}_F contient des puissances fractionnaires de ses éléments: si $L \in \mathcal{L}_F$ est un opérateur d'ordre $n \neq 0$ et si p est un nombre entier, il existe un opérateur $M \in \mathcal{L}_F$ d'ordre p tel que $M^n = L^p$. Si l'équation (1.1) admet une algèbre \mathcal{A}_F infinie, l'algèbre \mathcal{L}_F est non triviale et détermine la partie linéaire principale de \mathcal{A}_F [96]. Si \mathcal{L}_F contient des éléments rationnels en D , on peut les utiliser pour la construction récurrente d'algèbres \mathcal{A}_F .

§ 2 Transformations d'équivalence

On considère des transformations de Lie- Bäcklund dans \mathcal{A} , c'est-à-dire celles déterminées par une substitution $(x, u) \mapsto (y, v)$, soit:

$$y = Y(x, u, u_1, \dots, u_n), \quad v = V(x, u, u_1, \dots, u_n), \quad (2.1)$$

qui donne naissance par les équations:

$$D_x = D_x(Y)D_y, \quad (2.2)$$

$$D_x(V) = v_1 D_x(Y), \quad v_{i+1} = D_y(v_i); \quad \frac{dV}{dt} = v_t + v_1 \frac{dY}{dt} \quad (2.3)$$

et:

$$D_x(Y)\tilde{f} = (D_x(Y)V_* - D_x(V)Y_*)f, \quad f \in \mathcal{A}, \quad (2.4)$$

aux applications $D_x \mapsto D_y$, $u_i \mapsto v_i$, $f \mapsto \tilde{f}$.

Une équation:

$$v_t = H(y, v, v_1, \dots, v_m) \quad (2.5)$$

est dite équivalente à l'équation (1.1) s'il existe une transformation (2.1) du système dynamique (1.2) en (2.5). La transformation $v(t, y) \mapsto u(t, x)$ réciproque de (2.1) est en général une transformation de Bäcklund dans \mathcal{A} .

Théorème. Si les équations (1.1) et (2.5) sont équivalentes les algèbres \mathcal{L}_F

et \mathcal{L}_H (donc \mathcal{A}_F et \mathcal{A}_H) sont isomorphes. L'isomorphisme $\mathcal{L}_F \leftrightarrow \mathcal{L}_H$ est déterminé par la relation:

$$\tilde{L}(V_* - v_1 Y_*) = (V_* - v_1 Y_*)L, \quad v_1 = \frac{D_x(V)}{D_x(Y)}, \quad (2.6)$$

où $L \in \mathcal{L}_F$, $\tilde{L} \in \mathcal{L}_H$.

Preuve. On utilise le fait qu'une équation de Lax $L_t = [A, L]$ est invariante par la transformation $\tilde{L} = MLM^{-1}$, $\tilde{A} = MAM^{-1} + M_t M^{-1}$ avec $M \in \mathcal{L}$ quelconque:

$$\tilde{L}_t - [\tilde{A}, \tilde{L}] = M(\tilde{L}_t - [A, L])M^{-1}.$$

§ 3 Exemples

1° Pour l'équation linéaire:

$$u_t = u_2, \quad (3.1)$$

on obtient aisément deux solutions indépendantes d'ordre 1:

$$L_1 = D, \quad L_2 = 2tD + x, \quad (3.2)$$

de l'équation (1.5). L'algèbre \mathcal{L}_F se compose des séries formelles en L_1 , L_2 à coefficients constants.

Les opérateurs (3.2) donne naissance par (2.6) aux solutions de (1.5) pour l'équation de Burgers $v_t = v_2 + vv_1$; car cette équation est équivalente à (3.1) par la transformation $v = V(u, u_1) = 2u^{-1}u_1$. Dans ce cas

$$V_* = 2(u^{-1}D - u^{-2}u_1) = 2Du^{-1}, \quad V_*^{-1} = \frac{1}{2}uD^{-1}$$

et (2.6) donne:

$$\tilde{L}_1 = Du^{-1}DuD^{-1} = D + Du^{-1}u_1D^{-1} = D + \frac{1}{2}DvD^{-1} = D + \frac{v}{2} + \frac{v_1}{2}D^{-1},$$

$$\tilde{L}_2 = 2tL_1 + Dx D^{-1} = 2tD + (x + tv) + (1 + tv_1)D^{-1}.$$

2° La distribution de la chaleur dans l'hydrogène cristallin est conduite par l'équation non linéaire [156]:

$$v_t = v^2 v_2, \quad v = v(t, y), \quad (3.3)$$

qui admet une algèbre de Lie-Bäcklund infinie ([25], [96]) et qui se réduit à (3.1) par une transformation de Bäcklund (cf. [156], [25]). Trouvons

par (2.6) une transformation (2.1) de (3.1) en (3.3). Après la substitution $z = v^{-1}$, $t \mapsto -t$, (3.3) devient $z_t = D_y(z^{-2}z_1)$ et introduisant un potentiel w ($z = w_1$), on obtient l'équation:

$$w_t = w_1^{-2}w_2, \quad (3.4)$$

équivalente à (3.3). Pour (3.4) on trouve une solution simple $L_o = D_y w_1^{-1}$ de l'équation (1.5) et la relation (2.6) avec $L = L_o$, $\tilde{L} = L_1 = D_x$ donne l'équivalence (3.4) \leftrightarrow (3.1) ponctuelle:

$$x = w, \quad u = y. \quad (3.5)$$

A l'aide de (2.3) on obtient donc la transformation:

$$y = u, \quad v = u_1, \quad (3.6)$$

de (3.1) en (3.3). Pour (3.6)

$$V_* - v_1 Y_* = u_1 D_x u_1^{-1}, \quad (V_* - v_1 Y_*)^{-1} = u_1 D_x^{-1} u_1^{-1}$$

et (3.2), (2.6), (2.2) donnent des solutions (rationnelles en D) de l'équation (1.5) pour (3.3):

$$\begin{aligned} \tilde{L}_1 &= u_1 D_x u_1^{-1} D_x u_1 D_x^{-1} u_1^{-1} = v^2 D_y^2 v D_y^{-1} v^{-2}, \\ \tilde{L}_2 &= 2t \tilde{L}_1 + u_1 D_x x D_x^{-1} u_1^{-1} = (2tv^2 D_y^2 + 1) v D_y^{-1} v^{-2} + x. \end{aligned}$$

§ 4 Proposition

L'équation

$$u_t = u^3 u_3, \quad u = u(t, x), \quad (4.1)$$

est équivalente à l'équation de Korteweg-de Vries:

$$v_t = v_3 + v v_1, \quad v = v(t, y). \quad (4.2)$$

Preuve. Après la transformation:

$$v = 3w_1^{-1}w_3 - \frac{9}{2} w_1^{-2}w_2^2,$$

l'équation (4.2) devient:

$$w_t = w_3 - \frac{3}{2} w_1^{-1}w_2^2, \quad w = w(t, y) \quad (4.3)$$

et par $x = w$, $u = w_1$ on réalise l'application (4.3) \mapsto (4.1).

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Paper 20

Group classification of equations of nonlinear filtration

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We consider the equation

$$u_t = v(u_x)u_{xx}, \quad (1)$$

describing the motion of a non-Newtonian, weakly compressible fluid in a porous medium with a nonlinear filtration law

$$V = - \int v(u_x) du_x, \quad (2)$$

where V is the speed of filtration and u is the pressure. The classification of filtration laws on the bases of symmetry properties of equation (1) relative to groups of point transformations is of interest for solution of applied problems. Such a problem has already been considered in [106] in connection with the study of translational motions of a nonlinear, viscoplastic fluid. Some special cases of a group extension were indicated in [106] (namely, $v = \exp(u_x)$ and $v = u_x^\sigma$), but the complete solution of the problem of group classification was not given.

In the present paper the problem of group classification is solved, and a large new class of filtration laws with extended symmetry is discovered. It is

also shown that equation (1) possesses a larger group of equivalence transformations than the nonlinear heat equation $w_t = (f(w)w_x)_x$ connected with equation (1) by the substitution $u_x = w$.

§ 1 Solution of the determining equation

At the first step group classification assumes the solution of the determining equation of the group admitted by the differential equation in question [144]. Analysis of the determining equation in the present case leads to the following results.

1) For an arbitrary filtration coefficient $v(u_x)$ equation (1) admits a four-dimensional algebra with the basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.$$

2) If the function v satisfies the differential equation of first order

$$\frac{v'}{v} = \frac{a + 2bu_x}{c + du_x - bu_x^2} \quad (3)$$

with constants a, b, c, d , then an extension of the group occurs, and the following operator is added to the foregoing:

$$X_5 = -at \frac{\partial}{\partial t} + bu \frac{\partial}{\partial x} + (cx + du) \frac{\partial}{\partial u}. \quad (4)$$

3) In the case

$$\frac{v'}{v} = \frac{2a}{au_x + b} \quad (5)$$

a further extension of the group occurs. Equation (1) here admits an infinite group similar to the group of the linear heat equation. It is known ([86], § 19.4, Example 3) that if in (5) $b = 0$, $a \neq 0$, the corresponding equation (1) is mapped to the heat equation $\bar{u}_t = \bar{u}_{\bar{x}\bar{x}}$ by the change of variables

$$\bar{x} = u, \quad \bar{u} = x. \quad (6)$$

It is easy to see that actually all equations satisfying the condition (5) reduce to the linear heat equation.

§ 2 Equivalence transformations

At the next step of the classification equivalence, transformations are used, i.e., nondegenerate changes of the variables t, x, u taking any equation of the form (1) into an equation of the same form, generally speaking, with a different filtration coefficient $v(u_x)$. To clarify the physical meaning of the filtration laws obtained after equivalence transformations it is expedient to determine also the corresponding transformations of the filtration speed V . The set of all equivalence transformations forms a group E . We shall first find a continuous subgroup E_c of it using the infinitesimal method [144].

We seek an operator of the group E_c

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu \frac{\partial}{\partial v} + \theta \frac{\partial}{\partial V}$$

from the invariance condition of equations (1), (2) written as the system

$$u_t = v u_{xx}, \quad v_t = 0, \quad v_x = 0, \quad v_u = 0, \quad v_{u_t} = 0,$$

$$V_{u_x} = v, \quad V_t = 0, \quad V_x = 0, \quad V_u = 0, \quad V_{u_t} = 0.$$

Here u, v and V are considered as differential variables [86]: u on the space (t, x) and v and V on the extended space (t, x, u, u_t, u_x) . The coordinates ξ^1, ξ^2, η of the operator Y are sought as functions of t, x, u , while the coordinates μ and θ are sought as functions of t, x, u, u_t, u_x, v, V . By solving the corresponding determining equations, we obtain the operators

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial u}, \quad Y_4 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v},$$

$$Y_5 = x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \quad Y_6 = u \frac{\partial}{\partial x} + 2u_x v \frac{\partial}{\partial v}, \quad Y_7 = x \frac{\partial}{\partial u}, \quad Y_8 = u \frac{\partial}{\partial u},$$

which generate the 8-parameter equivalence group

$$\begin{aligned} t' &= \alpha t + \gamma_1, & x' &= \beta_1 x + \beta_2 u + \gamma_2, & u' &= \beta_3 x + \beta_4 u + \gamma_3, \\ v' &= \frac{(\beta_1 + \beta_2 u_x)^2 v}{\alpha}, & V' &= \frac{BV}{\alpha}, & B &= \beta_1 \beta_4 - \beta_2 \beta_3 \end{aligned} \quad (7)$$

with coefficients

$$\begin{aligned} \alpha &= a_4, & \beta_1 &= a_5, & \beta_2 &= a_6, & \beta_3 &= a_5 a_7 a_8, & \beta_4 &= (1 + a_6 a_7) a_8, \\ \gamma_1 &= a_1 a_4, & \gamma_2 &= a_2 a_5 + a_3 a_6, & \gamma_3 &= (a_3 + a_2 a_5 a_7 + a_3 a_6 a_7) a_8, \end{aligned} \quad (8)$$

where a_i is the parameter of the one-parameter subgroup with generator Y_i (therefore, $a_4, a_5, a_8 > 0$).

It is easy to see that the reflections $t \mapsto -t$ and $x \mapsto -x$ are also contained in the group E . Adding them to the continuous group E_c , we obtain transformations (7) whose coefficients $\alpha, \beta_i, \gamma_i$ are chosen not from conditions (8) but can assume arbitrary values satisfying the condition of non-degeneracy ($\alpha \neq 0, B \neq 0$). It can be proved that the transformations (7) obtained in this way provide the complete equivalence group E . Among the transformations of E/E_c the transformation (6), obtained by rotation in the (x, u) -plane and reflection in x , will be of special interest to us.

§ 3 Results of the classification

Let us apply the equivalence transformations for the final classification of equations (1) with the additional operator (4). It can be shown that under these transformations the condition (3) goes over into a similar relation written in terms of the coefficients a', b', c', d' connected with a, b, c, d by the equations

$$\begin{aligned} a &= Ba' + 2\beta_3\beta_4b' - 2\beta_1\beta_2c' - 2\beta_2\beta_3d', & b &= \beta_4^2b' - 2\beta_2^2c' - \beta_2\beta_4d' \\ c &= \beta_1^2c' - \beta_3^2b' + \beta_1\beta_3d', & d &= 2\beta_1\beta_2c' - 2\beta_3\beta_4b' + (\beta_1\beta_4 + \beta_2\beta_3)d'. \end{aligned} \quad (9)$$

It follows from (9) that the discriminant $\Delta = d^2 + 4bc$ under the action of (7) undergoes the conformal transformation $\Delta' = B^{-2}\Delta$. Therefore the set of functions $v(u_x)$ satisfying (3) decomposes into three equivalence classes in accordance with whether $\Delta = 0$, $\Delta > 0$ or $\Delta < 0$.

1) If $\Delta = 0$, then all equations of this class are equivalent to the equation

$$u_t = \exp(u_x)u_{xx}. \quad (10)$$

2) If $\Delta > 0$, then any equation of this class can be mapped to the form

$$u_t = (u_x)^{\lambda-1}u_{xx}, \quad \lambda \geq 0, \quad (11)$$

where

$$\lambda = \frac{|\frac{a}{b} + \beta + \alpha|}{|\beta - \alpha|}$$

and α and β are the zeros of $c + dx - bx^2$.

3) If $\Delta < 0$, then any equation of this class reduces to the form

$$u_t = \frac{\exp(\lambda \arctan u_x)}{1 + u_x^2} u_{xx}, \quad \lambda \geq 0, \quad (12)$$

where

$$\lambda = \frac{2|a + d|}{\sqrt{-\Delta}}.$$

The corresponding equivalence transformation (7) can easily be found from equations (9).

Remark 20.1. The transformation (6) maps the equation

$$u_t = u_x^{\lambda-1} u_{xx}$$

to the equation

$$\bar{u}_t = \bar{u}_{\bar{x}}^{-(\lambda+1)} \bar{u}_{\bar{x}\bar{x}}.$$

In particular, the equation

$$u_t = \frac{u_{xx}}{u_x}$$

is invariant with respect to the transformation (6).

Remark 20.2. In contrast to (10) and (11), equation (12) corresponds to a filtration law “with saturation”, i.e. the speed of filtration V tends to a finite limit as $u_x \rightarrow \infty$.

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Paper 21

Group analysis of ODEs and the invariance principle in mathematical physics

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Preface

“The extraordinary significance of Lie’s work for the general development of geometry cannot be overstated. I am convinced that in years to come it will grow still greater” - so wrote Felix Klein in his nomination (see [100], pages 10-28) of S. Lie’s results on the group-theoretic foundations of geometry to receive the first N.I. Lobachevsky prize at the Kazan Imperial University in 1895. There can be no doubt that the work of Lie in differential equations merits equally high evaluation.

Marius Sophus Lie was born on 17 December 1842 at the vicarage in Nordfjordeid on the west coast of Norway. He was the sixth (youngest) child in the family of Johan Herman Lie, a Lutheran pastor. Starting in 1857 he studied in Christiania (now Oslo), first at a grammar school and then

* *Authors note to this 2006 edition:* The original manuscript contained more material, but it was shortened, for publication in *Uspekhi Matematicheskikh Nauk*, by technical reasons. Here, I used the original version of § 3.3.2. Besides, I added § 5.1 and the proof of Theorem 21.18 in § 4.4, both taken from the original manuscript.

during 1859–1865 at the University. Among other events that determined the choice of his scientific direction, of importance were:



Figure 1: Sophus Lie (1842-1899)

self-study in 1868 of the geometric works of Chasles, Poncelet, and Plücker, a journey to Germany and France (1869–1870) and a meeting there with Klein (which grew into a close friendship and a long term partnership), Chasles, Jordan, and Darboux. During 1872–1886 Lie worked at the University of Christiania, and from 1886 to 1898 at the University of Leipzig. Lie died on 18 February 1899 in Christiania. The life, development of ideas and work of the great Norwegian mathematician are discussed in detail in the excellent book *Sophus Lie* by E.M. Polishchuk (Nauka, Leningrad, 1983, in Russian). This book reflects in sufficient detail the main aspects of Lie's scientific work and enables the reader to get close to Lie the man*.

One of the more remarkable achievements of Lie was the discovery that the majority of known methods of integration of ordinary differential equations, which seemed up to that time artificial and internally disconnected, could be derived in a unified manner using the theory of groups (see the book [120]). Moreover, Lie [113], [114], [115], [116] provided a classification of all ordinary differential equations of arbitrary order in terms of the groups they admit, and thus described the whole collection of equations for which integration or reduction of order could be effected by group-theoretic considerations. However, these and other very valuable results he obtained (see [126]) could not for a long time be widely disseminated and remained known to only a few. It could be said that this is the state of affairs today with methods of solution of problems of mathematical physics: many of these are of a group-theoretic nature, but are presented as a result of a lucky guess.

The present survey, written to mark the coming 150th anniversary of Sophus Lie, was planned as a possible way of presenting group-theoretic methods in an advanced course on differential equations. It represents a significant expansion of the chapter on Group Analysis of the course on

* *Authors note to this 2006 edition:* Today, the reader can find a vivid description of Lie's life story and the complete bibliography of his works in the remarkable book [167].

Mathematical Physics I teach at Moscow Institute of Physics and Technology. In this work I present the basic methods of group analysis and results of Lie as well as related results of other authors including some of my own.

CHAPTER 1

Definitions and elementary applications

The concept of a differential equation has two components. In the case of first-order ordinary differential equations, for example, one has to

1) define the surface $F(x, y, y') = 0$ in the space of the three variables x, y, y' . I call this surface the *frame* (or *hull*) of the differential equation;

2) determine a *class of solutions*; for example, a smooth solution is a continuously differentiable function $\varphi(x)$ such that the curve $y = \varphi(x)$, $y' = \varphi'(x)$ belongs to the frame, i.e. $F(x, \varphi(x), \varphi'(x)) = 0$ identically for all x , where $\varphi'(x) = \frac{\partial \varphi(x)}{\partial x}$. Passage to discontinuous or generalized solutions (with the same frame) changes the situation drastically.

In the process of integrating differential equations, the crucial step is the simplification of the frame by a suitable change of variables. To this end we use the *symmetry group* of the equation (or the *admitted group*), which is defined as the group of transformations of the (x, y) -plane, the prolongation of which to the derivatives y', \dots leaves the frame of the equation under consideration invariant.

Example 21.1. The Riccati equation

$$y' + y^2 - 2/x^2 = 0$$

admits the following non-homogeneous group of dilations (stretchings):

$$G: \bar{x} = xe^a, \bar{y} = ye^{-a}$$

since the frame of this equation given in Fig.2 is invariant with respect to non-homogeneous dilations

$$\bar{x} = xe^a, \bar{y} = ye^{-a}, \bar{y}' = y'e^{-2a}$$

obtained by the prolongation of the group G to the first derivative y' . After the change of variables $t = \ln x$, $u = xy$, we have the equation

$$u' + u^2 - u - 2 = 0.$$

Its frame is a cylindrical surface (a parabolic cylinder given in Fig. 3) in accordance with the fact that the group of dilations is transformed into the group of translations $\bar{t} = t + a$, $\bar{u} = u$, $\bar{u}' = u'$.

In this chapter we concentrate mainly on finding and using one-parameter symmetry groups for ordinary differential equations.

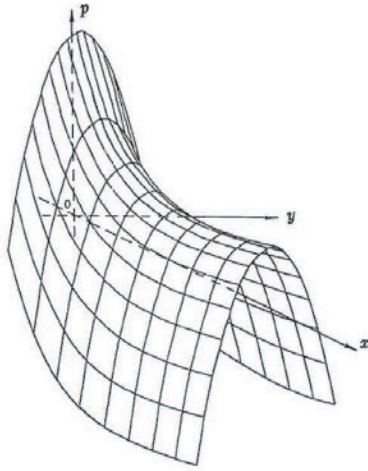


Figure 2: Frame of the equation $y' + y^2 - 2/x^2 = 0$, $p = y'$.

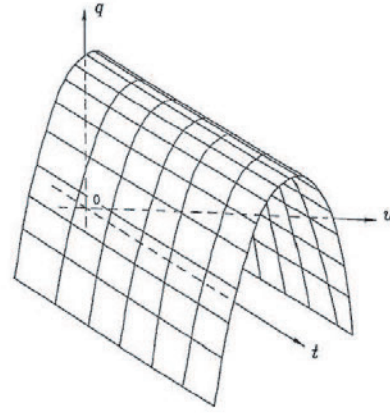


Figure 3: Frame of the equation $u' + u^2 - u - 2 = 0$, $q = u'$.

§ 1.1 One-parameter transformation groups

Consider invertible transformations in the (x, y) -plane

$$\bar{x} = \varphi(x, y, a), \quad \bar{y} = \psi(x, y, a), \quad (1.1)$$

depending on a real parameter a . It is assumed that

$$\varphi|_{a=0} = x, \quad \psi|_{a=0} = y. \quad (1.2)$$

These transformations are said to form a *one-parameter group* G if consecutive application of two transformations (1.1) is equivalent to the application of a third transformation of the form (1.1). By a suitable choice of the parameter a the group property can be written in the form

$$\begin{aligned} \varphi(\bar{x}, \bar{y}, b) &= \varphi(x, y, a + b), \\ \psi(\bar{x}, \bar{y}, b) &= \psi(x, y, a + b). \end{aligned} \quad (1.3)$$

If the conditions (1.3) hold not for all values of the parameters a and b (taken from some fixed interval) but only for sufficiently small values of these parameters, then G is called a *local one-parameter group*. In group analysis of differential equations one deals precisely with local groups, which for brevity are called simply groups.

Transformations (1.1) are called *point transformations* (unlike, for example, *contact transformations*, in which the transformed values \bar{x}, \bar{y} depend also on the derivative y' , see § 1.7), and the group G is called a *group of*

point transformations. It follows from equations (1.2), (1.3) that the inverse transformation to (1.1) is obtained by changing the sign of the group parameter a :

$$x = \varphi(\bar{x}, \bar{y}, -a), \quad y = \psi(\bar{x}, \bar{y}, -a). \quad (1.4)$$

Denoting by T_a the transformation (1.1) which takes a point (x, y) into a point (\bar{x}, \bar{y}) , by I the identity transformation, by T_a^{-1} the transformation inverse to T_a , which takes the point (\bar{x}, \bar{y}) into (x, y) , and by $T_b T_a$ a composition of two transformations, we summarize the properties (1.2)-(1.4):

Definition 21.1. The collection G of transformations T_a , is called a *local one-parameter group* if

$$1) T_0 = I \in G, \quad 2) T_b T_a = T_{a+b} \in G, \quad 3) T_a^{-1} = T_{-a} \in G$$

for sufficiently small a and b .

Let us expand the functions φ , ψ in Taylor series in the parameter a in a neighborhood of $a = 0$ and write down the infinitesimal transformation (1.1), with (1.2) taken into account, as

$$\bar{x} \approx x + \xi(x, y)a, \quad \bar{y} \approx y + \eta(x, y)a, \quad (1.1')$$

where

$$\xi(x, y) = \left. \frac{\partial \varphi(x, y, a)}{\partial a} \right|_{a=0}, \quad \eta(x, y) = \left. \frac{\partial \psi(x, y, a)}{\partial a} \right|_{a=0}. \quad (1.5)$$

The vector (ξ, η) with components (1.5) is the tangent vector (at the point (x, y)) to the curve described by the transformed points (\bar{x}, \bar{y}) , and is therefore called the *tangent vector field* of the group.

For example, for the group of rotations

$$\bar{x} = x \cos a + y \sin a, \quad \bar{y} = y \cos a - x \sin a$$

the infinitesimal transformation has the form

$$\bar{x} \approx x + ya, \quad \bar{y} \approx y - xa.$$

A one-parameter group is completely determined, provided that its infinitesimal transformation (1.1') is known, by solving the following *Lie equations* with the initial conditions:

$$\begin{aligned} \frac{d\varphi}{da} &= \xi(\varphi, \psi), & \varphi|_{a=0} &= x, \\ \frac{d\psi}{da} &= \eta(\varphi, \psi), & \psi|_{a=0} &= y. \end{aligned} \quad (1.6)$$

The tangent vector field (1.5) is often written as a first-order differential operator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \quad (1.7)$$

which behaves as a *scalar* under an arbitrary change of variables, unlike the *vector* (ξ, η) . Lie called the operator (1.7) the *symbol of the infinitesimal transformation* (1.1') or the *symbol of the group*. Then the term *infinitesimal operator* of the group (or, for short, the *operator of the group*) entered into usage. In physical literature one frequently encounters the term *generator of the group*.

Definition 21.2. A function $F(x, y)$ is called an *invariant of the group of transformations* (1.1) if for each point (x, y) the function F is constant on the trajectories described by the transformed points (\bar{x}, \bar{y}) :

$$F(\bar{x}, \bar{y}) = F(x, y).$$

Theorems 21.1–21.4 quoted below are well known. We omit the proofs of the first three, while the proof of the fourth one is included as it uses the method of passing to canonical variables we shall use in what follows.

Theorem 21.1. A function $F(x, y)$ is an invariant if and only if it satisfies the partial differential equation

$$XF \equiv \xi(x, y) \frac{\partial F}{\partial x} + \eta(x, y) \frac{\partial F}{\partial y} = 0. \quad (1.8)$$

It follows from Theorem 21.1 that every one-parameter group of point transformations in the plane has one independent invariant, which can be taken to be the left-hand side of the first integral $J(x, y) = C$ of the ordinary differential equation associated with (1.8) (the characteristic equation):

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}. \quad (1.8')$$

Any other invariant is then a function of J .

The concepts introduced above can be generalized in an obvious way to the multi-dimensional case where one considers, instead of transformations in the (x, y) -plane, groups of transformations

$$\bar{x}^i = f^i(x, a), \quad i = 1, \dots, n, \quad (1.9)$$

in the n -dimensional space of points $x = (x^1, \dots, x^n)$. Let us dwell on this higher-dimensional case and consider a system of equations

$$F(x) = 0, \dots, F_s(x) = 0, \quad s < n. \quad (1.10)$$

We assume that the rank of the matrix $\|\partial F_k / \partial x^i\|$ is equal to s at all points x satisfying the system (1.10). The system of equations (1.10) defines an $(n - s)$ -dimensional surface M .

Definition 21.3. The system of equations (1.10) is said to be *invariant with respect to the group G* of transformations (1.9) if each point x of the surface M moves along the surface, i.e. $\bar{x} \in M$ whenever $x \in M$.

Theorem 21.2. The system of equations (1.10) is invariant with respect to the group of transformations G (1.9) with the symbol

$$X = \xi^i(x) \frac{\partial}{\partial x^i}, \quad \xi^i(x) = \left. \frac{\partial f^i(x, a)}{\partial a} \right|_{a=0} \quad (1.11)$$

if and only if

$$XF_k|_M = 0, \quad k = 1, \dots, s. \quad (1.12)$$

Theorem 21.3. Let the system of equations (1.10) admit a group G and let the tangent vector $\xi(x)$ of the group do not vanish on the surface M defined by equations (1.10). Then the system (1.9) can be equivalently rewritten in the form

$$\Phi_k(J_1(x), \dots, J_{n-1}(x)) = 0, \quad k = 1, \dots, s, \quad (1.10')$$

where $J_1(x), \dots, J_{n-1}(x)$ is a *basis of invariants* (a set of all functionally independent invariants) of the group G . Equations (1.10) and (1.10') are equivalent in the sense that they define the same surface M .

Theorem 21.4. Every one-parameter group of transformations (1.1) reduces to the group of translations $\bar{t} = t + a$, $\bar{u} = u$ with the symbol $X = \frac{\partial}{\partial t}$ by a suitable change of variables

$$t = t(x, y), \quad u = u(x, y).$$

Such variables t, u are called *canonical variables*.

Proof. Under the change of variables $t = t(x, y)$, $u = u(x, y)$ the differential operator (1.7) transforms according to the formula

$$\bar{X} = X(t) \frac{\partial}{\partial t} + X(u) \frac{\partial}{\partial u}. \quad (1.13)$$

Therefore canonical variables are found from the equations

$$X(t) = 1, \quad X(u) = 0. \quad (1.14)$$

§ 1.2 Prolongation formulae

Let us work out the rules for the transformation of the derivatives y', y'' under the action of point transformations (1.1) regarded as a change of variables. It is convenient to use the *total differentiation*

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots$$

The derivatives are transformed according to the formulae

$$\bar{y}' \equiv \frac{d\bar{y}}{d\bar{x}} = \frac{D\psi}{D\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \equiv P(x, y, y', a), \quad (1.15)$$

$$\bar{y}'' \equiv \frac{d\bar{y}'}{d\bar{x}} = \frac{DP}{D\varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y}. \quad (1.16)$$

If one starts from a group G of point transformations (1.1) and extends its action to y' by adding the formula (1.15), one obtains the so-called *first prolongation* $G_{(1)}$ of G . The prolonged group $G_{(1)}$ acts in the space of three variables (x, y, y') . If one adds to (1.1) the formulae (1.15)-(1.16), one arrives at the twice prolonged group $G_{(2)}$ acting in the space (x, y, y', y'') .

Substituting into (1.15), (1.16) the infinitesimal transformation (1.1') $\bar{x} = x + a\xi$, $\bar{y} = y + a\eta$ and neglecting all terms of order $o(a)$, we obtain the infinitesimal transformations of derivatives:

$$\begin{aligned} \bar{y}' &= \frac{y' + aD(\eta)}{1 + aD(\xi)} \approx [y' + aD(\eta)][1 - aD(\xi)] \\ &\approx y' + [D(\eta) - y'D(\xi)]a \equiv y' + a\zeta_1, \end{aligned}$$

$$\begin{aligned} \bar{y}'' &= \frac{y'' + aD(\zeta_1)}{1 + aD(\xi)} \approx [y'' + aD(\zeta_1)][1 - aD(\xi)] \\ &\approx y'' + [D(\zeta_1) - y''D(\xi)]a \equiv y'' + a\zeta_2. \end{aligned}$$

Therefore the symbols of the prolonged groups $G_{(1)}$, $G_{(2)}$ are written

$$X_{(1)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial y'}, \quad \zeta_1 = D(\eta) - y'D(\xi), \quad (1.17)$$

$$X_{(2)} = X_{(1)} + \zeta_2 \frac{\partial}{\partial y''}, \quad \zeta_2 = D(\zeta_1) - y''D(\xi). \quad (1.18)$$

The operators (1.17) and (1.18) are termed the *first* and *second prolongations of the infinitesimal operator* (1.7), respectively. The term *prolongation*

formulae is frequently used to denote the expressions for the additional coordinates:

$$\zeta_1 = D(\eta) - y'D(\xi) = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y, \quad (1.19)$$

$$\begin{aligned} \zeta_2 = D(\zeta_1) - y''D(\xi) = & \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' \\ & + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''. \end{aligned} \quad (1.20)$$

§ 1.3 Groups admitted by differential equations

Let G be a group of point transformations and let $G_{(1)}$, $G_{(2)}$ be its first and second prolongations, respectively, defined in § 1.2.

Definition 21.4. They say that a first-order ordinary differential equation

$$F(x, y, y') = 0$$

admits a group G of point transformations (1.1) if the frame of the equation in question (that is, the two-dimensional surface defined by this equation in the three-dimensional space of the variables x, y, y') is invariant, in the sense of Definition 21.3, with respect to the prolonged group $G_{(1)}$.

Likewise, a second-order differential equation

$$F(x, y, y', y'') = 0 \quad (1.21)$$

admits a group G if its frame is invariant with respect to the twice prolonged group $G_{(2)}$. The frame of equation (1.21) is a three-dimensional surface in the space x, y, y', y'' .

This definition can be extended naturally to differential equations of higher order, as well as to systems of partial differential equations.

It is not hard to construct differential equations admitting a given group by using Theorem 21.3 on the representation of invariant equations in terms of invariants. Here it is useful to keep in mind that every one-parameter group of transformations in the (x, y) -plane has exactly one independent invariant. Prolongation the group to the first derivative y' adds one more invariant which necessarily depends on y' and is therefore called a *first-order differential invariant*. Likewise, a twice prolonged group has three functionally independent invariants, where the additional invariant depends on the second derivative and is called a *second-order differential invariant*.

It is convenient to compute second-order differential invariants by using the following theorem due to Lie ([120], Ch. 16, §5).

Theorem 21.5. Let an invariant $u(x, y)$ and a first-order differential invariant $v(x, y, y')$ of a group G be known. Then the derivative

$$w = \frac{dv}{du} = \frac{v_x + y'v_y + y''v_{y'}}{u_x + y'u_y} \equiv \frac{Dv}{Du}$$

is a second-order differential invariant of the group G .

Any (second or lower order) differential invariant of the group G is a function of u, v, w . By further differentiation we can obtain differential invariants of higher orders $d^2v/du^2, d^3v/du^3, \dots$.

For reference, we quote in Tables * 6.2 and 6.3 some first-order and second-order differential equations together with the symbol of the one-parameter group they admit. These tables have been constructed using Theorems 21.3 and 21.5.

§ 1.4 Integration and reduction of order using one-parameter groups

Group theory sheds light on a whole range of approaches and results concerning integration of specific types of equations that are widely used in practice. It allows us to understand better the interconnections between various particular methods of solution. In this section we discuss the simplest applications of one-parameter groups in problems of integration and reduction of the order of ordinary differential equations.

§ 1.4.1 Integrating factor

This method is applicable only to first-order equations. Let us consider a first-order ordinary differential equation written in a form symmetric with respect to x and y :

$$Q(x, y)dx - P(x, y)dy = 0. \quad (1.22)$$

Theorem 21.6. (Lie [109]). Equation (1.22)

$$Q(x, y)dx - P(x, y)dy = 0$$

admits a one-parameter group with the symbol (1.7)

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

*The notation Table 6. n means Table n in Chapter 6.

if and only if the function

$$\mu = \frac{1}{\xi Q - \eta P} \quad (1.23)$$

is an integrating factor for the equation (1.22) provided that $\xi Q - \eta P \neq 0$.

Example 21.2. Let us consider the Riccati equation

$$y' + y^2 = \frac{2}{x^2}. \quad (1.24)$$

Since it is an equation of power type, it is natural to expect that it admits a group of stretchings. Substituting $\bar{x} = ax$, $\bar{y} = by$, we have

$$\bar{y}' + \bar{y}^2 - \frac{2}{\bar{x}^2} = \frac{b}{a} y' + b^2 y^2 - \frac{2}{a^2 x^2}.$$

Equation (1.24) is invariant if $b/a = b^2 = 1/a^2$. Hence $b = 1/a$. It means that equation (1.24) admits the one-parameter group of stretchings (it can be written in the form $\bar{x} = xe^a$, $\bar{y} = ye^{-a}$) with the symbol

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \quad (1.25)$$

Writing equation (1.24) in the form (1.22),

$$dy + \left(y^2 - \frac{2}{x^2}\right)dx = 0 \quad (1.24')$$

and applying the formula (1.23), one obtains the integrating factor

$$\mu = \frac{x}{x^2 y^2 - xy - 2}.$$

After multiplication by this factor, (1.24') is brought to the following form:

$$\frac{xdy + (xy^2 - 2/x)dx}{x^2 y^2 - xy - 2} = \frac{xdy + ydx}{x^2 y^2 - xy - 2} + \frac{dx}{x} = d\left(\ln x + \frac{1}{3} \ln \frac{xy - 2}{xy + 1}\right) = 0.$$

Integration yields the following solution:

$$\frac{xy - 2}{xy + 1} = \frac{C}{x^3} \quad \text{or} \quad y = \frac{2x^3 + C}{x(x^3 - C)}.$$

§ 1.4.2 Method of canonical variables

If we know the admitted group, we can use Theorem 21.4 on reducing a one-parameter group to a group of translations by introducing canonical variables. Since *the property of invariance of an equation with respect to a group is independent on the choice of variables*, introduction of canonical variables leads to an equation which does not depend on one of the variables, and hence can be integrated in quadratures (in the case of a first-order equation) or admits reduction of order (in the case of a higher-order equation). The following examples clarify these statements.

Example 21.3. Let us solve the Riccati equation (1.24) by the method of canonical variables. The group with the generator (1.25) admitted by equation (1.24) is a particular case of a group of non-homogeneous stretchings from Table 6.1 for $k = -1$. In this case canonical variables are

$$t = \ln x, \quad u = xy.$$

Changing to these variables reduces (1.24) to a clearly integrable form

$$u' + u^2 - u - 2 = 0.$$

Example 21.4. The linear second-order equation

$$y'' + f(x)y = 0 \tag{1.26}$$

admits a group of stretchings in y with the operator

$$X = y \frac{\partial}{\partial y}. \tag{1.27}$$

Reduction to translations is effected by the change of variables $u = x$, $t = \ln y$, after which (1.26) takes the form

$$u'' - u' + f(u)u'^3 = 0.$$

The resulting equation does not contain the independent variable. Therefore its order can be reduced by the well-known substitution $u' = p(u)$, and the problem reduces to the integration of the Riccati equation

$$\frac{dp}{du} + f(u)p^2 - 1 = 0.$$

§ 1.4.3 Invariant differentiation

If a second-order or higher-order equation admits a one-parameter group, its order can be reduced by one by using Theorems 21.3 and 21.5. This method is an invariant formulation of well-known methods of reduction of order of equations that do not contain either the dependent or the independent variable explicitly.

For definiteness, let us dwell upon second-order equations. By Theorem 21.3 any second-order equation admitting a group G can be written in terms of the differential invariants u, v, w of, respectively, zeroth, first, and second order.

By Theorem 21.5 the second-order differential invariant can be chosen to be of the form $w = dv/du$ and the invariant second-order differential equation under consideration can be written as

$$\frac{dv}{du} = F(u, v). \quad (1.28)$$

Thus we can reduce the order: if we have found an integral

$$\Phi(u, v, C) = 0 \quad (1.29)$$

of the first-order equation (1.28), the solution of the original second-order equation can be reduced to quadratures. Indeed, substitution of known expressions $u(x, y)$ and $v(x, y, y')$ into (1.29) leads to a first-order differential equation, which admits the group G in view of the invariance of u and v , and which can therefore be integrated by quadratures.

Let us use this second method to reduce the order of the equation (1.26). Writing down the first prolongation of the operator (1.27)

$$X_{(1)} = y \frac{\partial}{\partial y} + y' \frac{\partial}{\partial y'},$$

we find the invariants $u = x$, $v = y'/y$. Theorem 21.5 yields the second-order differential invariant

$$\frac{dv}{du} = \frac{y''}{y} - \frac{y'^2}{y^2} = \frac{y''}{y} - v^2,$$

whence

$$\frac{y''}{y} = \frac{dv}{du} + v^2.$$

On substituting this expression into (1.26) one obtains a first-order equation of the form (1.28), namely the following Riccati equation:

$$\frac{dv}{du} + v^2 + f(u) = 0.$$

§ 1.5 Determining equation

Let us move on now to the problem of construction of an admitted group. Suppose we have a second-order equation (1.21) (first-order equations are included as a particular case, in which $F_{y''} = 0$). By Definition 21.4 and Theorem 21.2, the infinitesimal invariance test has the form of equation (1.12) applied to the twice prolonged operator $X_{(2)}$:

$$X_{(2)}F|_{F=0} \equiv (\xi F_x + \eta F_y + \zeta_1 F_{y'} + \zeta_2 F_{y''})|_{F=0} = 0, \quad (1.30)$$

where ζ_1 and ζ_2 are given by the prolongation formulae (1.19) and (1.20). Equation (1.30) is called the *determining equation* for the group admitted by the ordinary differential equation (1.21).

In what follows we shall consider differential equations written in the form solved for y'' :

$$y'' = f(x, y, y'). \quad (1.31)$$

In this case, after substituting the values of ζ_1 , ζ_2 from (1.19), (1.20) with y'' given by the right-hand side of (1.31), the determining equation (1.30) assumes the form

$$\begin{aligned} \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)f \\ - [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y]f_{y'} - \xi f_x - \eta f_y = 0. \end{aligned} \quad (1.32)$$

Here $f(x, y, y')$ is a known function since we are dealing with a *given* differential equation (1.31), whereas the coordinates ξ and η of the required admitted operator (1.7) are unknown functions of x, y . Since the left-hand side of (1.32) contains in addition to x, y also the quantity y' considered as an *independent variable*, the determining equation decomposes into several independent equations, thus becoming an overdetermined system of differential equations for ξ, η . Solving this system, we find all the operators admitted by the differential equation (1.31) under consideration.

Example 21.5. Let us find the operators

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

admitted by the second order-equation

$$y'' + \frac{1}{x}y' - e^y = 0. \quad (1.33)$$

Here $f = e^y - \frac{1}{x}y'$ and the determining equation (1.32) has the form

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)\left[e^y - \frac{y'}{x}\right]$$

$$+\frac{1}{x}[\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y] - \xi\frac{y'}{x^2} - \eta e^y = 0.$$

The left-hand side of this equation is a third-degree polynomial in the variable y' . Therefore the determining equation decomposes into the following four equations, obtained by setting the coefficients of the various powers of y' equal to zero:

$$(y')^3 : \quad \xi_{yy} = 0, \quad (1.34)$$

$$(y')^2 : \quad \eta_{yy} - 2\xi_{xy} + \frac{2}{x}\xi_y = 0, \quad (1.35)$$

$$y' : \quad 2\eta_{xy} - \xi_{xx} + \left(\frac{\xi}{x}\right)_x - 3\xi_y e^y = 0, \quad (1.36)$$

$$(y')^0 : \quad \eta_{xx} + \frac{1}{x}\eta_x + (\eta_y - 2\xi_x - \eta)e^y = 0. \quad (1.37)$$

The integration of Equations (1.34) and (1.35) with respect to y yields

$$\xi = p(x)y + a(x), \quad \eta = \left(p' - \frac{p}{x}\right)y^2 + q(x)y + b(x).$$

We substitute these expressions for ξ , η in Equations (1.36), (1.37) and observe that the functions ξ and η are polynomial in y , while the left-hand sides of Equations (1.36), (1.37) contain e^y . Since Equations (1.36), (1.37) should vanish identically in x and y , it follows that

$$\xi_y = 0, \quad \eta_y - 2\xi_x - \eta = 0.$$

These equations yield:

$$\xi = a(x), \quad \eta = -2a'(x).$$

Substituting these expressions into (1.36), we have

$$\left(a' - \frac{a}{x}\right)' = 0,$$

whence

$$a = C_1 x \ln x + C_2 x.$$

Then Equation (1.37) is satisfied identically.

Thus, the general solution to the determining equations (1.34)–(1.37) has the form

$$\xi = C_1 x \ln x + C_2 x, \quad \eta = -2[C_1(1 + \ln x) + C_2]$$

and contains two arbitrary constants C_1 and C_2 . In view of the linearity of the determining equations, the general solution can be represented as a linear combination of two independent solutions

$$\begin{aligned}\xi_1 &= x \ln x, & \eta_1 &= -2(1 + \ln x); \\ \xi_2 &= x, & \eta_2 &= -2.\end{aligned}$$

This means that equation (1.33) admits two linearly independent operators

$$X_1 = x \ln x \frac{\partial}{\partial x} - 2(1 + \ln x) \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y}, \quad (1.38)$$

and that the set of all admitted operators is a two-dimensional vector space spanned by (1.38).

§ 1.6 Lie algebras

Let us return now to general properties of determining equations. As can be seen from (1.32), a determining equation is a linear partial differential equation for the functions ξ and η in the variables x and y . Therefore the set of its solutions forms a vector space as it was already noted in Example 21.5. However, in addition there is another property which is a specific characteristic of determining equations. It turns out that the set of solutions of a determining equation forms a very special type of vector space called a *Lie algebra* (the term is due to H. Weyl; Lie himself referred to an *infinitesimal group*).

Definition 21.5. A *Lie algebra of operators* (1.7) is a vector space L with the following property. If the operators

$$X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}, \quad X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y} \quad (1.39)$$

are elements of L , so is their commutator

$$[X_1, X_2] = (X_1(\xi_2) - X_2(\xi_1)) \frac{\partial}{\partial x} + (X_1(\eta_2) - X_2(\eta_1)) \frac{\partial}{\partial y}. \quad (1.40)$$

The Lie algebra is denoted by the same letter L , and its dimension is the dimension of the vector space L .

A general statement about properties of determining equations for second-order ordinary differential equations is the following result due to Lie [120].

Theorem 21.7. The set of all solutions of a determining equation (1.32) for second-order equations (1.31) forms an algebra L_r of dimension $r \leq 8$. The maximal dimension $r = 8$ is reached if and only if Equation (1.31) either is linear or can be linearized by a change of variables.

§ 1.7 Contact transformations

In addition to point transformations (1.1), groups of *contact* or *tangent* transformations [125], [124] have been found to be of use in mechanics, geometry, and the theory of differential equations. We will discuss the multidimensional case with an arbitrary number n of independent variables $x = (x^1, \dots, x^n)$ and one dependent variable u . For a larger number of dependent variables there are no contact transformations that are not point transformations; a proof can be found, e.g. in [144], §28.3, [7], §9, or [86] §14.1.

Let us denote by u' the collection of first derivatives $u_i = \frac{\partial u}{\partial x^i}$ and consider a one-parameter group of transformations

$$\bar{x}^i = \varphi^i(x, u, u', a), \quad \bar{u} = \psi(x, u, u', a), \quad \bar{u}_i = \omega_i(x, u, u', a)$$

in the $(2n + 1)$ -dimensional space of variables (x, u, u') .

Theorem 21.8. An operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i}$$

is a symbol of a group of contact transformations if and only if

$$\xi^i = -\frac{\partial W}{\partial u_i}, \quad \eta = W - u_i \frac{\partial W}{\partial u_i}, \quad \zeta_i = \frac{\partial W}{\partial x^i} + u_i \frac{\partial W}{\partial u}$$

for some function $W = W(x, u, u')$.

Lie [125] calls the function W defined in this theorem the *characteristic function* of the group of contact transformations. Construction of a group of contact transformations admitted by an ordinary differential equation thus reduces to the determination of the characteristic function, the function $W = W(x, y, p)$ of three variables x , y and $p = y'$, by substitution into the determining equation of an operator of the form

$$X = -W_p \frac{\partial}{\partial x} + (W - pW_p) \frac{\partial}{\partial y} + (W_x + pW_y) \frac{\partial}{\partial p}. \quad (1.41)$$

Example 21.6. Let us find the operators (1.41) admitted by the equation

$$y''' = 0. \quad (1.42)$$

Prolongation of the operator (1.40) to the second and third derivatives has the form

$$X_{(3)} = X + \zeta_2 \frac{\partial}{\partial y''} + \zeta_3 \frac{\partial}{\partial y'''} ,$$

where ζ_2 is given by the *second prolongation formula* (1.20) and ζ_3 is given by the similar *third prolongation formula*:

$$\zeta_3 = D(\zeta_2) - y''' D(\xi).$$

The determining equation has the form

$$(X_{(3)}y''')|_{y'''=0} = \zeta_3|_{y'''=0} = 0$$

and decomposes into the following four equations:

$$\begin{aligned} (y'')^3 &: W_{ppp} = 0, \\ (y'')^2 &: W_{yp} + W_{xpp} + pW_{ypp} = 0, \\ y'' &: W_{xy} + W_{xyp} + pW_{yy} + 2pW_{xyp} + p^2W_{yyp} = 0, \\ (y'')^0 &: W_{xxx} + 3pW_{xxy} + 3p^2W_{xyy} + p^3W_{yyy} = 0. \end{aligned}$$

The above system of differential equations for W can be easily and yields:

$$\begin{aligned} W &= C_1 + C_2x + C_3x^2 + C_4y + C_5p + C_6xp + C_7(x^2p - 2xy) \\ &+ C_8p^2 + C_9(xp^2 - 2yp) + C_{10}(x^2p^2 - 4xyp + 4y^2). \end{aligned}$$

The function W depends on ten arbitrary constants C_i . Consequently, equation (1.42) admits a ten-parameter group of contact transformations. The Lie algebra of this group is obtained by substituting the function W in the formula (1.41) and has the following basis:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial y}, \quad X_5 = \frac{\partial}{\partial x}, \\ X_6 &= x \frac{\partial}{\partial x}, \quad X_7 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_8 = 2p \frac{\partial}{\partial x} + p^2 \frac{\partial}{\partial y}, \\ X_9 &= (y - xp) \frac{\partial}{\partial x} - \frac{1}{2} xp^2 \frac{\partial}{\partial y} - \frac{1}{2} p^2 \frac{\partial}{\partial p}, \\ X_{10} &= \left(xy - \frac{1}{2} x^2 p\right) \frac{\partial}{\partial x} + \left(y^2 - \frac{1}{4} x^2 p^2\right) \frac{\partial}{\partial y} + \left(yp - \frac{1}{2} xp^2\right) \frac{\partial}{\partial p}. \end{aligned}$$

The operators X_1, \dots, X_7 generate a 7-dimensional algebra of a group of point transformations. This is the maximal group of point transformations admitted by equation (1.42). The prolongations of the operators X_1, \dots, X_7 to the derivative $y' = p$ contained in (1.41) are, naturally, omitted here. Three remaining operators, X_8, X_9, X_{10} , are symbols of proper contact transformations.

This seemingly simple example has a deep geometrical meaning. Namely, Lie established a remarkable connection between the group of contact transformations for equation (1.42) and conformal mappings in three-dimensional space (see [125], Ch. 10, §2).

Remark 21.1. For every second-order differential equation there are infinitely many contact transformations transforming it into any other given second-order equation, for example into $y'' = 0$. In particular, every second-order ordinary differential equation admits an infinite group of contact transformations. The situation changes drastically for higher-order equations. In this connection, in addition to the example considered above, it is useful to keep in mind that for a third-order equation to be reducible to the simplest form $y''' = 0$ by a contact transformation it is necessary that it be of the form*

$$y''' + A(y'')^3 + B(y'')^2 + Cy'' + D = 0$$

with coefficients A, B, C, D depending only on x, y, y' ([125], Ch. 3, §3).

Lie also indicated connections between contact transformations and the theory of waves and was groping towards a group-theoretic approach to the Huygens principle (see [124] and the book [125], p. 97), from which Fig. 3 is taken). According to Lie, Huygens' construction of the wave front is identical with the statement that the contact transformations

$$\bar{x} = x + \frac{tp}{\sqrt{1+p^2}}, \quad \bar{y} = y - \frac{t}{\sqrt{1+p^2}}, \quad \bar{p} = p$$

defined by the characteristic function $W = -\sqrt{1+p^2}$ form a one-parameter group with the group parameter t .

Development of this idea of Lie together with the above-mentioned relation of contact and conformal transformations has become subsequently the basis of the solution of the problem on the Huygens principle (in the sense of Hadamard) for equations with a non-trivial conformal group. See [77] (Paper 10 in this volume) or [86], Ch. 2, as well as the survey [95].

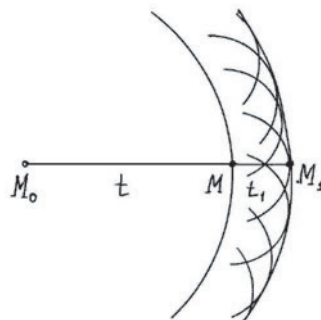


Figure 4: Construction of the wave front.

* *Authors note to this 2006 edition:* All third-order equations reducible to a linear equation by point and contact transformations are determined in [94].

CHAPTER 2

Integration of second-order equations admitting a two-dimensional algebra

“I noticed that the majority of ordinary differential equations which were integrable by the old methods were invariant under certain transformations, and that the integration methods consisted in using that property. Having thus represented variety of old integration methods from a general viewpoint, I set myself a natural problem: to develop a general theory of integration for all ordinary differential equations admitting finite or infinitesimal transformations.” (S. Lie [117], Preface, page IV).

§ 2.1 Consecutive reduction of order

We know from § 1.4 that one can reduce the order of a second-order equation once by using an admitted one-dimensional algebra. See also [10], [56] (Chapter II, Section IV: Infinitesimal transformations), [97], [38], [138], [134], [26], [165], [87]. Therefore it is natural to expect that if there is a two-dimensional algebra, one can reduce the order twice, i.e. one can integrate the equation. Let us start with a simple example.

§ 2.1.1 An instructive example

Let us consider the following linear equation:

$$y'' + y' - \frac{y}{x} = 0. \quad (2.1)$$

Clearly, it has the solution $y = x$ and by the superposition principle it admits the group of transformations $\bar{y} = y + ax$ with symbol $X_1 = x \frac{\partial}{\partial y}$. In addition, equation (2.1) is homogeneous and therefore admits the group of stretchings in y with symbol $X_2 = y \frac{\partial}{\partial y}$. Thus, for equation (2.1) we know at least two admitted operators (out of the eight operators possible by Theorem 21.7):

$$X_1 = x \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}. \quad (2.2)$$

Their commutator is

$$[X_1, X_2] = X_1. \quad (2.3)$$

Therefore the vector space with basis (2.2) is a two-dimensional Lie algebra.

Let us take the first prolongation of the operator X_1 :

$$X_{1(1)} = x \frac{\partial}{\partial y} + \frac{\partial}{\partial y'},$$

Its invariants are $u = x$, $v = y' - \frac{y}{x}$. By Theorem 21.5 we have a second-order differential invariant

$$\frac{dv}{du} = y'' - \frac{y'}{x} + \frac{y}{x^2} = y'' - \frac{v}{u},$$

from which we have $y'' = \frac{dv}{du} + \frac{v}{u}$. Expressing the left-hand side of (2.1) in the variables u, v , we obtain the following first-order equation (1.28):

$$\frac{dv}{du} + \left(1 + \frac{1}{u}\right)v = 0. \quad (2.4)$$

Next let us find the action of the operator X_2 in the (u, v) -plane. To that end, we prolong it to y' and pass to the variables u, v according to (1.13):

$$X_{2(1)} = y \frac{\partial}{\partial y} + y' \frac{\partial}{\partial y'} \mapsto Y = X_{2(1)}(u) \frac{\partial}{\partial u} + X_{2(1)}(v) \frac{\partial}{\partial v}.$$

We have $X_{2(1)}(u) = 0$, and hence obtain the generator $Y = v \frac{\partial}{\partial v}$ of a group of dilations *admitted by equation (2.4)*. Therefore *the operator X_1 has enabled us to reduce the order of equation (2.1) and to transform it to the form (2.4)*, while *the operator X_2 ensures the ability to further integrate* the resulting first-order equation. Construction of the solution of the original equation is done in reverse order: first we find the solution of equation (2.4):

$$v = \frac{C_1}{u} e^{-u},$$

which, by substituting $v = y' - \frac{y}{x}$, is rewritten as a first-order non-homogeneous equation

$$y' = \frac{y}{x} + \frac{C_1}{x} e^{-x}.$$

Above we started the reduction of order using the first of the operators (2.2). This was a completely random choice. Let us work out next whether anything changes if we start with the operator X_2 . Its differential invariants are

$$u = x, \quad v = \frac{y'}{y}, \quad \frac{dv}{du} = \frac{y''}{y'} - \frac{y'^2}{y^2}.$$

Using these, equation (2.1) assumes the form

$$\frac{dv}{du} + v^2 + v - \frac{1}{u} = 0. \quad (2.5)$$

Unlike equation (2.4), this Riccati equation does not have an obviously integrable form. Let us see if the so far unused operator X_1 provides us

with the simplifying change of variables. To that end we find how it acts in the (u, v) -plane. We have

$$X_{1(1)} = x \frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \mapsto Y = \left(\frac{1}{y} - \frac{xy'}{y^2} \right) \frac{\partial}{\partial v} = \frac{1}{y} (1 - uv) \frac{\partial}{\partial v}.$$

Since $y = e^{\int v dx}$, the resulting operator can be written in the form

$$Y = e^{\int v du} (1 - uv) \frac{\partial}{\partial v}. \quad (2.6)$$

This is not an operator of a group of point transformations, as it contains an integral term, but rather an operator of *non-local symmetry* of equation (2.5). If we try to find a simplifying change of variables by reducing (2.6) to a translation operator in accordance with § 1.4.2, it becomes clear that *the idea of starting the reduction of order of equation (2.1) using the operator X_2 is not a good one. Why?*

In order to understand the situation arising here and to formulate a general principle of integration using a known group, we need some facts about the structure of Lie algebras.

§ 2.1.2 Solvable Lie algebras

Let L_r be a finite-dimensional Lie algebra of dimension r and let N be a linear subspace in L_r .

Definition 21.6. A subspace N is called a *subalgebra* if $[X, Y] \in N$ for all $X, Y \in N$ (that is, if this subspace is a Lie algebra in its own right) and an *ideal* of the algebra L_r if $[X, Y] \in N$ for all $X \in N$ and all $Y \in L_r$.

If N is an ideal, then we can introduce an equivalence relation in the algebra L_r : operators X and Y in L_r are said to be equivalent if $Y - X \in N$. The set of all operators equivalent to a given operator X is called the *coset* represented by X ; every element Y of this coset has the form $Y = X + Z$ for some $Z \in N$. The cosets form a Lie algebra, which is called the *quotient algebra* of the L_r , with respect to the ideal N and is denoted by L_r/N . As elements of the quotient algebra we can take representatives of the corresponding cosets.

If all the constructions are to be considered in the complex domain, then we have the following theorem.

Theorem 21.9. Any algebra L_r , $r > 2$, contains a two-dimensional subalgebra. Moreover, every operator $X \in L_r$ can be included in a two-dimensional algebra.

Definition 21.7. An algebra L_r is called *solvable* if there exists a sequence

$$L_r \supset L_{r-1} \supset \cdots \supset L_1 \quad (2.7)$$

of subalgebras of dimensions $r, r-1, \dots, 1$ respectively, in which each subalgebra L_{s-1} is an ideal in L_s ($s = 2, \dots, r$).

Definition 21.8. Let X_1, \dots, X_r be a basis of the algebra L_r . The subspace spanned by the commutators $[X_\mu, X_\nu]$ of all possible pairs of basis operators forms an *ideal*, denoted by L'_r , and is called the *derived algebra*. Derived algebras of higher order are defined recursively: $L_r^{(n+1)} = (L_r^{(n)})'$, $n = 1, 2, \dots$.

Theorem 21.10. An algebra L_r is solvable if and only if a derived algebra of L_r of some order is identically zero: $L_r^{(n)} = 0$ for some $n > 0$.

Corollary 21.1. Every two-dimensional algebra is solvable.

In the case of a two-dimensional algebra L_2 , to construct the sequence (2.7) we have to choose the basis X_1, X_2 so that the equality $[X_1, X_2] = \alpha X_1$ holds. Then the one-dimensional algebra L_1 spanned by X_1 forms an ideal in L_2 , while the quotient algebra L_2/L_1 can be identified with the one-dimensional algebra spanned by X_2 .

Now we can give the answer to the question posed at the end of § 2.1.1. As can be seen from the equality (2.3), the operator X_1 , generates an ideal L_1 in the algebra L_2 with the basis (2.2). Having reduced the order of equation (2.1) using this ideal L_1 , we have obtained a first-order equation (2.4) which admits the quotient algebra L_2/L_1 (which we identified with the one-dimensional algebra spanned by X_2) with a natural definition of its action on the (u, v) -plane. But when, so as to reduce the order, we used the one-dimensional algebra with basis X_2 , which is not an ideal, we lost this additional symmetry. In our attempt to restore the symmetry, we found it necessary to extend the concept of symmetry by introducing the non-local symmetry operator (2.6). Therefore, if we want to deal exclusively with point symmetries, reduction of order must be done using ideals, which is what Lie himself did.

Theorem 21.9 and the corollary of Theorem 21.10 clarify why two-dimensional algebras (or, equivalently, two-parameter groups) play a fundamental role in problems of integration of second-order equations. It is rather obvious that if we want to integrate an equation of order $n > 2$ by the method of consecutive reduction of order, the equation should admit a solvable n -dimensional Lie algebra. For a detailed discussion of this question see, for example, [134], Theorems 2.60, 2.61, and 2.64. Here we restrict ourselves to the case of second-order equations.

§ 2.2 The method of canonical variables

If a second-order equation admits a two-dimensional Lie algebra, then instead of consecutive reduction of order one can directly make a change of variables x, y (introduce *canonical variables* in the sense of Lie), in which the equation under consideration assumes an integrable form. By Theorem 21.9 this also includes equations that admit Lie algebras of higher dimensions (in this case complex-valued changes of variables may be necessary). Furthermore, there is a special method for integrating second-order equations admitting a three-dimensional algebra (see [120], Ch. 24) but we do not discuss it here.

§ 2.2.1 Changes of variables and of a basis in an algebra

First we consider structural peculiarities of two-dimensional Lie algebras.

One of the properties, namely solvability of every two-dimensional algebra, was already mentioned in the corollary of Theorem 21.10. The property of solvability is, obviously, invariant with respect to changes of variables x, y and is independent of the choice of a basis.

Let us now discuss two other invariant properties, which lead to a partition of all two-dimensional Lie algebras into four types and form the basis of the method of canonical variables for integration using two-dimensional algebras. These properties are related to the rule of transformation of the *commutator* (1.40) of the operators (1.39) and of their *pseudo-scalar product*

$$X_1 \vee X_2 = \xi_1 \eta_2 - \eta_1 \xi_2 \quad (2.8)$$

under a non-degenerate change of variables

$$t = t(x, y), \quad u = u(x, y); \quad \frac{\partial(t, u)}{\partial(x, y)} \equiv t_x u_y - t_y u_x \neq 0 \quad (2.9)$$

and a change of a basis in L_2 :

$$X'_1 = \alpha_1 X_1 + \alpha_2 X_2, \quad X'_2 = \beta_1 X_1 + \beta_2 X_2; \quad \Delta \equiv \alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0. \quad (2.10)$$

Lemma 21.1. When changing to a new basis (2.10) the commutator of the operators (1.39) transforms according to

$$[X'_1, X'_2] = \Delta [X_1, X_2],$$

while under a change of variables (2.9) it transforms covariantly (see (1.13)), i.e.

$$[\overline{X}_1, \overline{X}_2] = \overline{[X_1, X_2]}.$$

Proposition 21.1. In every two-dimensional Lie algebra we can choose a basis X_1, X_2 so that either of the two following commutation relations holds:

$$[X_1, X_2] = 0 \quad \text{or} \quad [X_1, X_2] = X_1.$$

Both relations are invariant with respect to the change of variables (2.9).

Lemma 21.2. The pseudo-scalar product (2.8) transforms under (2.9) and (2.10) according to

$$\overline{X}_1 \vee \overline{X}_2 = \frac{\partial(t, u)}{\partial(x, y)} (X_1 \vee X_2), \quad X'_1 \vee X'_2 = \Delta (X_1 \vee X_2).$$

Proposition 21.2. The equation

$$X_1 \vee X_2 = 0$$

is invariant under any change of variables (2.9) and any change of a basis (2.10).

§ 2.2.2 Canonical forms of two-dimensional algebras

Propositions 21.1 and 21.2 lead to the following statements concerning the partition of all two-dimensional Lie algebras into four main types.

Theorem 21.11. Any two-dimensional Lie algebra can be reduced, by choosing an appropriate basis X_1, X_2 , to one of the four different types defined by the following structure relations:

- I. $[X_1, X_2] = 0, \quad X_1 \vee X_2 \neq 0;$
- II. $[X_1, X_2] = 0, \quad X_1 \vee X_2 = 0;$
- III. $[X_1, X_2] = X_1, \quad X_1 \vee X_2 \neq 0;$
- IV. $[X_1, X_2] = X_1, \quad X_1 \vee X_2 = 0.$

These structure relations are invariant under the change of variables (2.9).

Using the invariance of the structure relations I-IV with respect to the change of variables x, y one can simplify, by an appropriate change of variables, the form of basic operators of the two-dimensional algebras of the above types and arrive at the following result.

Theorem 21.12. A basis of any algebra L_2 can be reduced by a suitable change of variables (2.9) to one of the following forms:

$$\begin{aligned} \text{I. } X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}; \\ \text{II. } X_1 &= \frac{\partial}{\partial y}, & X_2 &= x \frac{\partial}{\partial y}; \\ \text{III. } X_1 &= \frac{\partial}{\partial y}, & X_2 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}; \\ \text{IV. } X_1 &= \frac{\partial}{\partial y}, & X_2 &= y \frac{\partial}{\partial y}. \end{aligned}$$

The corresponding variables x, y are called *canonical variables*.

§ 2.2.3 Integration algorithm

Let us find the second-order ordinary differential equations admitting the canonical two-dimensional Lie algebras of four indicated types and integrate them.

TYPE I. To construct all second-order equations admitting the Lie algebra L_2 with the basis $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y}$, we have to find a basis of second-order differential invariants of these operators. In this case the prolongations of the operators coincide with the operators themselves, so that the desired differential invariants are y' and y'' . Therefore, the general second-order equation admitting the algebra L_2 of the first type has the form

$$y'' = f(y').$$

It can be integrated by two quadratures:

$$\int \frac{dy'}{f(y')} = x + C_1 \quad \text{or explicitly} \quad y' = \varphi(x + C_1),$$

$$\text{whence} \quad y = \int \varphi(x + C_1) d(x + C_1) + C_2.$$

TYPE II. A basis of differential invariants of the algebra L_2 of type II is given by x and y' , and hence the invariant differential equation has the form

$$y'' = f(x).$$

Its solution is given by two quadratures:

$$y = \int \left(\int f(x) dx \right) dx + C_1 x + C_2.$$

TYPE III. The invariant equation has the form

$$y'' = \frac{1}{x} f(y').$$

It is also solved by two quadratures:

$$\int \frac{dy'}{f(y')} = \ln x + C_1$$

or explicitly $y' = \varphi(\ln x + C_1)$, whence $y = \int \varphi(\ln x + C_1) dx + C_2$.

TYPE IV. Here we have the invariant equation

$$y'' = f(x)y'$$

with the general solution

$$y = C_1 \int e^{\int f(x) dx} dx + C_2.$$

The results of Theorems 21.11 and 21.12, as well as the corresponding invariant equations, are given in Table 6.4, while the integration algorithm is presented in Table 6.6.

§ 2.2.4 An example of implementation of the algorithm

Let us apply the group algorithm to the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}. \quad (2.11)$$

1st step. Determination of the admitted algebra. Solving the determining equation (1.32) we find that

$$\xi = C_1 x^2 + C_2 x, \quad \eta = \left(C_1 x + \frac{C_2}{2} \right) y.$$

Hence equation (2.11) admits the Lie algebra L_2 with the basis

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y}.$$

According to Table 6.6 one can proceed directly to the third step.

3rd step. *Determination of the type of the algebra L_2 .* We have

$$[X_1, X_2] = -X_1, \quad X_1 \vee X_2 = -\frac{x^2 y}{2} \neq 0.$$

Therefore the algebra L_2 is of type III in Table 6.4. For the correspondence to be complete, one can change the sign of the operator X_2 and obtain the basis

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = -x \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial y} \quad (2.12)$$

obeying the structure relations of type III in Table 6.4.

4th step. *Determination of the integrating change of variables.* Here X_1 is the operator of the group of projective transformations of Table 6.1. Therefore we use the change of variables given there and, interchanging t and u , obtain:

$$t = \frac{y}{x}, \quad u = -\frac{1}{x}. \quad (2.13)$$

After this change of variables, the operators (2.12) become

$$\bar{X}_1 = \frac{\partial}{\partial u}, \quad \bar{X}_2 = \frac{t}{2} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}.$$

The difference between them and the corresponding operators of type III in Table 6.4 (the factor $1/2$ in \bar{X}_2) is of no importance. Excluding the solutions

$$y = Kx \quad (K = \text{const.}) \quad (2.14)$$

of equation (2.11), when $t = \text{const.}$, we rewrite equation (2.11) in the form

$$\frac{u''}{u'^2} + \frac{1}{t^2} = 0.$$

Whence, integrating once, we obtain

$$u' = \frac{t}{C_1 t - 1}.$$

If $C_1 = 0$ then

$$u = -\frac{t^2}{2} + C. \quad (2.15)$$

If $C_1 \neq 0$ then

$$u = \frac{t}{C_1} + \frac{1}{C_1^2} \ln |C_1 t - 1| + C_2. \quad (2.16)$$

5th step. *Solution in the original variables.* Substituting into (2.15), (2.16) the values (2.13) of the variables t , u and taking into account the isolated solution (2.14), we obtain the following general solution to equation (2.11):

$$y = Kx, \quad y = \pm\sqrt{2x + Cx^2}, \quad (2.17)$$

$$C_1y + C_2x + x \ln \left| C_1 \frac{y}{x} - 1 \right| + C_1^2 = 0. \quad (2.18)$$

CHAPTER 3

Group classification of second-order equations

“In a short report to the Scientific Society of Göttingen (3 December 1874) I presented, *inter alia*, an enumeration of all continuous transformation groups in two variables x, y and specially emphasized that this might be used as a basis for classification and for a rational theory of integration of all differential equations $f(x, y, y', \dots, y^{(m)}) = 0$ admitting a continuous group of transformations. Subsequently I carried out in detail the great programme outlined there.” (S. Lie [113], p. 187)

Below I present the results of the realization of that programme as they pertain to second-order equations. The restriction to second-order equations is not due to any limitation of the method but to the desire to concentrate on specific material and to provide exhaustive results.

§ 3.1 Equations admitting three-dimensional algebras

The group classification of ordinary differential equations is based on a listing of all possible Lie algebras of operators (1.7) on the (x, y) -plane. The basis operators of all the algebras are maximally simplified by means of a suitable change of variables (2.14). Algebras related by such a change of variables are called *similar*. Enumeration of algebras is made up to similarity, since equations admitting similar algebras are *equivalent* in the sense that they can be reduced to each other by a change of variables.

Lie’s group classification [113], [114], [115], [116] has a particularly simple form in the case of second-order equations of interest to us (the classification result for second-order equations is presented briefly and clearly in [118], §3). In this case the admitted algebra (by which we mean the maximal algebra admitted by a given equation) can be only of dimension 0,

1, 2, 3, or 8. Dimension 0 means that the equation does not admit a group of point transformations; examples of such equations are

$$y'' = e^{y'} + xy, \quad y'' = (y')^2 + xy, \quad y'' = 6y^2 + x, \quad y'' = 2y^3 + xy + C.$$

Dimensions 1 and 2 were considered in the previous chapters. Therefore we start with a discussion of equations admitting three-dimensional algebras.

§ 3.1.1 Classification in the complex domain

Lie performed his classification in the complex domain, using, where necessary, complex changes of variables or of algebra basis. Enumeration of all non-similar (with respect to a complex change of variables) three-dimensional algebras in the plane, derivation of the corresponding invariant second-order differential equations and methods of their solution, are all discussed in detail in the book [120], Part V. There, in Ch. 22, §3 and Ch. 24, §3 Lie gives a summary of the results of classification of three-dimensional algebras and the invariant equations. These results are reproduced here in Table 6.8. Let us see how the equations admitting given three-dimensional algebras are constructed in that table.

For all three-dimensional algebras the construction of an invariant equation is the same. It requires the solution of the determining equation (1.32) with respect to the unknown function $f(x, y, y')$ for known coordinates ξ and η of basis operators of the algebra L_3 under consideration. Therefore it suffices to demonstrate the construction for one of the algebras. As such an example, let us choose the second algebra from Table 6.8, i.e. the algebra with the basis

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}. \quad (3.1)$$

For the operator X_1 we have $\xi = 1$, $\eta = 1$. On substituting these values of ξ , η , the determining equation (1.32) assumes the form

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0,$$

whence

$$f = f(x - y, y').$$

Now one has to substitute into the determining equation the obtained value of f and the coordinates $\xi = x$, $\eta = y$ of the second of the operators (3.1). As a result, one obtains the equation $zf_z + f = 0$, where $z = x - y$. The solution of this equation has the form

$$f = \frac{g(y')}{x - y} \quad (3.2)$$

with an arbitrary function $g(y')$. Finally, the substitution into (1.32) of the function f of (3.2) and of the coordinates $\xi = x^2$, $\eta = y^2$ of the operator X_3 leads to the following equation for the function $g(y')$:

$$2y' \frac{dg}{dy'} - 3g + 2(y'^2 - y') = 0.$$

It follows that

$$g = -2(y' + Cy'^{3/2} + y'^2), \quad C = \text{const.} \quad (3.3)$$

Formulae (3.2), (3.3) provide the desired right-hand side of the invariant equation. Thus, the algebra L_3 with the basis (3.1) is admitted by the equation

$$y'' + 2 \frac{y' + Cy'^{3/2} + y'^2}{x - y} = 0$$

with an arbitrary constant C .

§ 3.1.2 Classification over the reals. Isomorphism and similarity

Lie's complex classification is quite sufficient from the point of view of integration methods. However, it is also of interest to consider the results of the classification over the reals, since we are dealing with real valued differential equations. The results pertaining to second-order equations with a three-dimensional algebra are summarized here. They supplement Lie's complex classification. Certain details can be found in [127].

Table 6.7 gives a perception of the difference between classifications in the real and the complex domains. It contains the well-known enumeration, due to Bianchi, of all possible structures of three-dimensional Lie algebras that are non-isomorphic over the reals (see [22] or, e.g. [148], §10 and [41], §24). In the table, the values of three different commutators of the basis, which define the structure of each algebra L_3 , are provided.

Recall that two algebras are said to be *isomorphic* if one can choose bases for the algebras such that the structure constants of the two algebras in these bases are equal. One can see from Table 6.7 that real non-isomorphic algebras can turn out to be isomorphic over the field of complex numbers (that is, when complex transformations of bases of algebras are allowed).

Note that in the case of two-dimensional Lie algebras there are precisely two non-isomorphic structures (both over the reals and over the complex numbers), which can be taken to be $[X_1, X_2] = 0$ and $[X_1, X_2] = X_1$ (see Table 6.4).

The concept of isomorphism is independent of a particular realization of a Lie algebra, e.g. as an algebra of vector fields (or of differential operators

(1.7)). In the case of Lie algebras of operators (1.7) there is an additional way of transforming one algebra into another, namely, by means of a change of variables x, y . Algebras related by a change of variables are called *similar*.

Similar algebras are isomorphic (see [144], §7.9). *The converse, in general, is not true* as can be seen from Tables 6.4 and 6.8. It is precisely similarity (and not simply isomorphism) that is of use in group analysis as a test for reducibility of one differential equation to another by a suitable change of variables. Nonetheless, establishing isomorphism is important as a first (necessary) step for the determination of similarity of algebras.

It is both interesting and useful to give more detailed instructions for the construction of similarity transformations. The main theorems on similarity of groups and Lie algebras were obtained by Lie ([117], p. 356) and Eisenhart* ([44], [45]). Here, we will dwell upon one of them. Let us first introduce the necessary notation.

Let L_r and \bar{L}_r be two r -dimensional Lie algebras of operators of the form (1.11) in the n -dimensional space of variables $x = (x^1, \dots, x^n)$. It was already mentioned above that a necessary condition for Lie algebras to be similar is that they are isomorphic. Therefore we assume that the algebras L_r and \bar{L}_r are isomorphic, their operators are written in the variables x and \bar{x} , respectively, and that their bases

$$X_\alpha = \xi_\alpha^i \frac{\partial}{\partial x^i}, \quad \bar{X}_\alpha = \bar{\xi}_\alpha^i \frac{\partial}{\partial \bar{x}^i} \quad (\alpha = 1, \dots, r)$$

are chosen so that the structure constants of the two algebras in these bases are equal. We are interested in the case when $r > n$ and the ranks of the matrices $\|\xi_\alpha^i\|$, $\|\bar{\xi}_\alpha^i\|$ are the same and equal to n . Renumbering the indices, if necessary, we shall assume that this is the value of the rank of the square matrices $\|\xi_h^i\|$, $\|\bar{\xi}_h^i\|$ ($i, h = 1, \dots, n$). Then the following conditions hold:

$$\xi_p^i = \varphi_p^h \xi_h^i, \quad \bar{\xi}_p^i = \bar{\varphi}_p^h \bar{\xi}_h^i \quad (i = 1, \dots, n; p = n+1, \dots, r), \quad (3.4)$$

where φ_p^h , $\bar{\varphi}_p^h$ are functions of x and \bar{x} , respectively, and where we sum over h from 1 to n . Under these assumptions, the following theorem is valid.

Theorem 21.13. For algebras L_r and \bar{L}_r with the same structure and the same number of variables x^i , \bar{x}^i to be similar, it is necessary and sufficient that the functions φ_p^h , $\bar{\varphi}_p^h$ satisfy the system of equations

$$\bar{\varphi}_p^h(\bar{x}) = \varphi_p^h(x) \quad (h = 1, \dots, n; p = n+1, \dots, r), \quad (3.5)$$

*Eisenhart used the term “equivalent groups”. We have used “similar groups” (and “similar algebras”) in view of what is said in the first paragraph of § 3.1. Two different words are used in the Russian text for “similar” and “equivalent”. (Ed.)

and that this system does not lead to any relations either among the variables x^i or among the variables \bar{x}^i .

Sketch of the proof. Similarity of the algebras under consideration means that there is a non-degenerate change of variables $\bar{x}^i = f^i(x)$ such that

$$\bar{\xi}_\alpha^i = \xi_\alpha^k \frac{\partial \bar{x}^i}{\partial x^k} \quad (i = 1, \dots, n; \alpha = 1, \dots, r). \quad (3.6)$$

Equations (3.6) and (3.4) yield $n(r - n)$ equations (3.5).

Necessity. If the algebras L_r and \bar{L}_r are similar, then, clearly, equations (3.5) are compatible. Therefore, elimination of \bar{x}^i from these equations does not lead to any relation between the variables x^i , and vice versa.

Sufficiency. For $\alpha = 1, \dots, n$ let us write the equations (3.6) in the explicit form

$$\frac{\partial \bar{x}^i}{\partial x^k} = \bar{\xi}_h^i(\bar{x}) \xi_k^h(x) \quad (i, k = 1, \dots, n) \quad (3.6')$$

where $\|\xi_k^h\|$ is the square matrix inverse to $\|\xi_h^i\|$, i.e.

$$\xi_k^h \xi_h^i = \delta_k^i, \quad \xi_i^h \xi_l^i = \delta_l^h \quad (i, k, h, l = 1, \dots, n).$$

One can show, using the conditions (3.5), that the system (3.6'), and hence the system (3.6), is compatible (a detailed treatment of this non-trivial step in the argument can be found in [45], §22). This means that there is a change of variables that establishes the similarity of the algebras L_r and \bar{L}_r .

Determination of the change of variables. To determine the similarity transformation one has to find \bar{x}^i by solving the mixed system of equations comprising functional equations (3.5) and differential equations (3.6') which are integrable in view of (3.5).

Example 21.7. Let L_3 and \bar{L}_3 be the first and the second algebras of Table 6.8. They are complex isomorphic. Indeed, taking for \bar{L}_3 the basis

$$\bar{X}_1 = \frac{1}{2}(X_1 + X_3), \quad \bar{X}_2 = iX_2, \quad \bar{X}_3 = \frac{i}{2}(X_1 - X_3),$$

one can easily verify that the two algebras have the same structure and belong to the type IX in Table 6.7. Thus, let us consider the three-dimensional Lie algebras L_3 and \bar{L}_3 in the plane with the same structure having the bases

$$X_1 = (1 + x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad X_3 = xy \frac{\partial}{\partial x} + (1 + y^2) \frac{\partial}{\partial y}$$

and

$$\bar{X}_1 = \frac{1}{2} \left[(1 + \bar{x}^2) \frac{\partial}{\partial \bar{x}} + (1 + \bar{y}^2) \frac{\partial}{\partial \bar{y}} \right], \quad \bar{X}_2 = i \left(\bar{x} \frac{\partial}{\partial \bar{x}} + \bar{y} \frac{\partial}{\partial \bar{y}} \right),$$

$$\overline{X}_3 = \frac{i}{2} \left[(1 - \bar{x}^2) \frac{\partial}{\partial \bar{x}} + (1 - \bar{y}^2) \frac{\partial}{\partial \bar{y}} \right],$$

respectively. Equations (3.4) written in the form

$$\xi_3^k = \sum_{h=1}^2 \varphi^h \xi_h^k, \quad \bar{\xi}_3^k = \sum_{h=1}^2 \bar{\varphi}^h \bar{\xi}_h^k,$$

yield:

$$\begin{aligned} xy &= (1 + x^2)\varphi^1 - y\varphi^2, & \frac{i}{2}(1 - \bar{x}^2) &= \frac{1}{2}(1 + \bar{x}^2)\bar{\varphi}^1 + i\bar{x}\bar{\varphi}^2, \\ 1 + y^2 &= xy\varphi^1 + x\varphi^2, & \frac{i}{2}(1 - \bar{y}^2) &= \frac{1}{2}(1 + \bar{y}^2)\bar{\varphi}^1 + i\bar{y}\bar{\varphi}^2. \end{aligned}$$

Hence we have

$$\varphi^1 = \frac{y}{x}, \quad \varphi^2 = \frac{1}{x} \quad \text{and} \quad \bar{\varphi}^1 = i \frac{1 + \bar{x}\bar{y}}{1 - \bar{x}\bar{y}}, \quad \bar{\varphi}^2 = -\frac{\bar{x} + \bar{y}}{1 - \bar{x}\bar{y}}.$$

With these functions φ^h and $\bar{\varphi}^h$ the system (3.5) is written

$$i \frac{1 + \bar{x}\bar{y}}{1 - \bar{x}\bar{y}} = \frac{y}{x}, \quad -\frac{\bar{x} + \bar{y}}{1 - \bar{x}\bar{y}} = \frac{1}{x}$$

and furnishes the change of variable

$$x = -\frac{1 - \bar{x}\bar{y}}{\bar{x} + \bar{y}}, \quad y = -i \frac{1 + \bar{x}\bar{y}}{\bar{x} + \bar{y}}$$

establishing the similarity transformation of the algebras L_3 and \overline{L}_3 in the complex plane. In this example, in order to find the change of variables, it was enough to solve equations (3.5).

Example 21.8. Let us consider the algebras L_3 and \overline{L}_3 with the bases

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

and

$$\overline{X}_1 = \frac{\partial}{\partial \bar{x}}, \quad \overline{X}_2 = \bar{x} \frac{\partial}{\partial \bar{x}} + \frac{\bar{y}}{2} \frac{\partial}{\partial \bar{y}}, \quad \overline{X}_3 = \bar{x}^2 \frac{\partial}{\partial \bar{x}} + \bar{x}\bar{y} \frac{\partial}{\partial \bar{y}},$$

respectively. They have the same structure (see Table 6.8). Equations (3.4), written as in the previous example, have the form

$$\begin{aligned} x^2 &= \varphi^1 + x\varphi^2, & \bar{x}^2 &= \bar{\varphi}^1 + \bar{x}\bar{\varphi}^2, \\ y^2 &= \varphi^1 + y\varphi^2, & \bar{x}\bar{y} &= \frac{1}{2} \bar{y}\bar{\varphi}^2. \end{aligned}$$

Hence

$$\varphi^1 = -xy, \quad \varphi^2 = x + y, \quad \bar{\varphi}^1 = -\bar{x}^2, \quad \bar{\varphi}^2 = 2\bar{x}.$$

With these functions φ^h and $\bar{\varphi}^h$, equations (3.5) assume the form

$$\bar{x}^2 = xy, \quad 2\bar{x} = x + y,$$

whence, eliminating \bar{x} , one obtains the relation

$$x - y = 0$$

By Theorem 21.13 the existence of this relation between x and y shows that these algebras L_3 and \bar{L}_3 are not similar (neither over the reals, nor over the field of complex numbers), though they are isomorphic.

§ 3.2 The general classification result

It can be shown (e.g. by inspecting all non-similar Lie algebras of all possible dimensions and the corresponding invariant equations) that if a second-order equation admits a Lie algebra of a dimension greater than 4, then it admits an 8-dimensional algebra. All such equations are linearizable (§ 1.6, Theorem 21.7) and equivalent to the equation $y'' = 0$ (see § 3.3.2, Remark 21.4). Thus, second-order ordinary differential equations can admit Lie algebras of the dimensions 1, 2, 3, and 8. The classification result is given in Table 6.9. See also [103].

All equations of Table 6.9, apart from those in the first line, and all equations obtained from them by an arbitrary change of variables (2.9) can be integrated by quadratures by means of the group method. Therefore the set of such equations is infinite and depends on four arbitrary constants, two arbitrary functions of one variable (see Table 6.9), and on arbitrary functions of two variables which arise as a result of the change of variables (2.9).

§ 3.3 Two remarkable classes of equations

The classes of equations considered in this section naturally arise from the previous classification. They deserve to be studied more thoroughly than it has been done up till now.

§ 3.3.1 Equation $y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0$. Linearization test

In practical situations of solving differential equations it is useful to have simple tests for linearization. Such tests, obtained by Lie, are summarized in the following theorem.

Theorem 21.14. The following statements are equivalent.

(1) A second-order ordinary differential equation

$$y'' = f(x, y, y') \quad (3.7)$$

is linearizable by a change of variables (2.9);

(2) equation (3.7) admits an 8-dimensional Lie algebra;

(3) equation (3.7) has the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0 \quad (3.8)$$

with coefficients F_3, F_2, F_1, F satisfying the compatibility conditions of the following auxiliary system:

$$\begin{aligned} \frac{\partial z}{\partial x} &= z^2 - Fw - F_1z + \frac{\partial F}{\partial y} + FF_2, \\ \frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3} \frac{\partial F_2}{\partial x} + \frac{2}{3} \frac{\partial F_1}{\partial y}, \\ \frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3} \frac{\partial F_1}{\partial y} + \frac{2}{3} \frac{\partial F_2}{\partial x}, \\ \frac{\partial w}{\partial y} &= -w^2 + F_2w + F_3z + \frac{\partial F_3}{\partial x} - F_1F_3; \end{aligned} \quad (3.9)$$

(4) equation (3.7) admits a two-dimensional Lie algebra with a basis X_1, X_2 such that

$$X_1 \vee X_2 = 0. \quad (3.10)$$

Example 21.9. Let us verify that the equation

$$y'' + F(x, y) = 0, \quad (3.11)$$

where the function F is non-linear in y , cannot be linearized.

Indeed, in this case equations (3.9) have the form

$$\begin{aligned} z_x &= z^2 - Fw + F_y, & z_y &= -zw, \\ w_x &= zw, & w_y &= -w^2. \end{aligned} \quad (3.12)$$

One of the compatibility conditions, namely $z_{xy} - z_{yx} = 0$, leads to $F_{yy} = 0$. Therefore for any function $F(x, y)$ that is non-linear in y the system (3.12) is incompatible.

Example 21.10. Let us single out all linearizable equations of the form

$$y'' = \frac{1}{x}f(y'). \quad (3.13)$$

In accordance with Theorem 21.14, one has to restrict himself to the equations

$$y'' + \frac{1}{x}(A_3 y'^3 + A_2 y'^2 + A_1 y' + A_0) = 0, \quad (3.14)$$

where the coefficients A_i are constant since the function f depends only on y' .

For equation (3.14), the auxiliary system (3.9) is written

$$\begin{aligned} z_x &= z^2 - \frac{A_0}{x}w - \frac{A_1}{x}z + \frac{A_0 A_2}{x^2}, & z_y &= -zw + \frac{A_0 A_3}{x^2} + \frac{A_2}{3x^2}, \\ w_x &= zw - \frac{A_0 A_3}{x^2} - \frac{2}{3} \frac{A_2}{x^2}, & w_y &= -w^2 + \frac{A_2}{x}w + \frac{A_3}{x}z - \frac{A_3}{x^2} - \frac{A_1 A_3}{x^2}. \end{aligned}$$

The condition $z_{xy} = z_{yx}$ yields $A_2(2 - A_1) + 9A_0 A_3 = 0$. Likewise, using the condition $w_{xy} = w_{yx}$ we obtain $3A_3(1 + A_1) - A_2^2 = 0$. These equations, upon setting $A_3 = -a$, $A_2 = -b$, yield:

$$A_1 = -\left(1 + \frac{b^2}{3a}\right), \quad A_0 = -\left(\frac{b}{3a} + \frac{b^3}{27a^2}\right).$$

Thus, we have the following result (see [127] and the references therein).

Equation (3.13) is linearizable if and only if it has the form

$$y'' = \frac{1}{x} \left[ay'^3 + by'^2 + \left(1 + \frac{b^2}{3a}\right)y' + \frac{b}{3a} + \frac{b^3}{27a^2} \right]$$

with arbitrary parameters a and b .

§ 3.3.2 Equations $y'' + a(x)y' + b(x)y = c(x)y^{-3}$

Consider the equation

$$y'' = ky^{-3}, \quad k = \text{const.}, \quad (3.15)$$

from Table 6.9 and replace there y'' by the general linear second-order differential expression with variable coefficients. Then we divide the resulting equation by the coefficient of the second-order derivative and obtain the equation

$$y'' + a(x)y' + b(x)y = c(x)y^{-3}. \quad (3.16)$$

Its particular case,

$$y'' + b(x)y = ky^{-3}, \quad k = \text{const.}, \quad (3.17)$$

has long been discussed in the literature [48], [151], [19], [155] as an example of a non-linear equation with exceptional properties. These properties are as follows.

First of all, it was noticed by Ermakov [48] that the general solution to equation (3.17) can be expressed via solutions of the corresponding linear equation

$$y'' + b(x)y = 0. \quad (3.17')$$

Namely, the solution of equation (3.17) with arbitrary initial data $y(x_0) = y_0 \neq 0$, $y'(x_0) = y_1$ is given by

$$y(x) = [u^2(x) + kW^{-2}v^2(x)]^{1/2}, \quad (3.18)$$

where u and v are two linearly independent solutions of equation (3.17') such that $u(x_0) = y_0$, $u'(x_0) = y_1$, $v(x_0) = 0$, $v'(x_0) \neq 0$, and W is the Wronskian: $W = uv' - vu' = \text{const.} \neq 0$. Since equation (3.17) is reducible to the linear equation, it satisfies a "superposition principle", but of course in a non-linear form (see [155], pp. 154, 389). This is the second remarkable property of equation (3.17). Equation (3.17) is also of importance in applications. It appears in the study of travelling wave type solutions of the Schrödinger equation [28], in the theory of elasticity (see [155], §3.3), etc.

The peculiarities of equation (3.17) mentioned above are related to the fact that, for an arbitrary coefficient $b(x)$ and constant $k \neq 0$, it admits a three-parameter group, and, moreover, reduces to equation (3.15) by a change of variables. The latter equation is easily integrated with the help of invariant solutions. Now, using the group-theoretical approach, I generalize the class of equations (3.17) and single out from (3.16) the equations having the properties of (3.17).

Theorem 21.15. Equation (3.16) admits the following Lie algebras:

$$\begin{aligned} (i) \quad & L_8 \quad \text{if } c(x) = 0, \\ (ii) \quad & L_3 \quad \text{if } c(x) = k \exp(-2 \int a(x) dx), \quad k = \text{const.} \neq 0, \\ (iii) \quad & L_0 \quad \text{in all other cases.} \end{aligned} \quad (3.19)$$

Proof. If $c(x) = 0$, equation (3.16) is linear and the statement (i) follows from Theorem 21.7 in § 1.6. Let $c(x) \neq 0$. Considering the determining equation (1.32) for the operators

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (3.20)$$

admitted by equation (3.16) and proceeding as in Example 21.5 from § 1.5, we obtain

$$\xi = \xi(x), \quad \eta = \frac{1}{2} [\xi'(x) - a(x)\xi(x)] y \quad (3.21)$$

together with the following two equations for determining $\xi(x)$:

$$[c'(x) + 2a(x)c(x)]\xi = 0 \quad (3.22)$$

and

$$\xi''' + p(x)\xi' + \frac{1}{2}p'(x)\xi = 0, \quad (3.23)$$

where $p = 4b - 2a' - a^2$. Equation (3.22) shows that either

$$c'(x) + 2a(x)c(x) = 0 \quad (3.24)$$

or $\xi = 0$. Integration of equation (3.24) implies that $c(x)$ has the form (3.19). Furthermore, (3.23) is an ordinary differential equation of the third order, and hence its solution $\xi(x)$ depends on three arbitrary constants. This proves the statement (ii). It remains to note that if $\xi = 0$ then the second equation (3.21) yields $\eta = 0$, and hence (iii). This completes the proof of the theorem.

Remark 21.2. The proof of Theorem 21.15 shows that the calculation of the three-dimensional Lie algebra admitted by the non-linear equation

$$y'' + a(x)y' + b(x)y = \frac{k}{y^3} e^{-2 \int a(x) dx} \quad (3.25)$$

with a constant $k \neq 0$ and arbitrary functions $a(x)$ and $b(x)$ requires only integration of the linear third-order equation (3.23). Multiplying the latter equation by ξ and integrating one obtains

$$\xi\xi'' - \frac{1}{2}\xi'^2 + \frac{1}{2}p(x)\xi^2 = \text{const.}$$

whence, denoting $\xi = w^2$ and substituting $p = 4b - 2a' - a^2$, one has

$$w'' + \left(b - \frac{1}{2}a' - \frac{1}{4}a^2\right)w = \frac{l}{w^3}, \quad l = \text{const.} \quad (3.26)$$

According to (3.21), the coordinates of the admitted operators are expressed via solution of equation (3.26) as follows:

$$\xi = w^2, \quad \eta = \left(ww' - \frac{1}{2}aw^2\right)y. \quad (3.27)$$

Furthermore, it suffices to integrate the linear equation obtained from (3.26) by letting $l = 0$. Taking its two linearly independent solutions, $w_1(x)$ and $w_2(x)$, and substituting them in (3.27) one obtains two operators, X_1 and X_2 , respectively, admitted by equation (3.25). The commutator $X_3 = [X_1, X_2]$ furnishes the third operator (3.20) admitted by equation (3.25).

Example 21.11. Consider equation (3.15), $y'' = ky^{-3}$. Here $a = b = 0$, and equation (3.26) with $l = 0$ has the form $w'' = 0$. Taking its two linearly independent solutions, $w_1 = 1$ and $w_2 = x$, we obtain from (3.27) $\xi_1 = 1$, $\eta_1 = 0$ and $\xi_2 = x^2$, $\eta_2 = xy$, respectively. The corresponding operators (3.20) are:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}.$$

These operators together with their commutator

$$X_3 = [X_1, X_2] = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

provide the basis of L_3 given in Table 6.9.

Example 21.12. Consider the equation

$$y'' + y = ky^{-3}. \quad (3.28)$$

Here $a = 0$, $b = 1$, and equation (3.26) with $l = 0$ has the form $w'' + w = 0$. Its two linearly independent solutions, $w_1 = \cos x$ and $w_2 = \sin x$, the following operators (3.20) admitted by equation (3.28):

$$X_1 = \cos^2 x \frac{\partial}{\partial x} - y \cos x \sin x \frac{\partial}{\partial y}, \quad X_2 = \sin^2 x \frac{\partial}{\partial x} + y \cos x \sin x \frac{\partial}{\partial y}.$$

Their commutator

$$X_3 = [X_1, X_2] = \sin(2x) \frac{\partial}{\partial x} + y \cos(2x) \frac{\partial}{\partial y}$$

provides the third operator admitted by equation (3.28).

In investigating any family of equations, it is significant to know *equivalence transformations* of the family of equations in question, i.e. transformations mapping any individual equation of the family into an equation of the same family. Consider, in particular, the family of equations of the form (3.16). Then an equivalence transformation is a change of variables $\bar{x} = \varphi(x, y)$, $\bar{y} = \psi(x, y)$ such that any equation $y'' + a(x)y' + b(x)y = c(x)y^{-3}$, upon rewriting in the new variables, has again the form (3.16):

$$\frac{d^2 \bar{y}}{d\bar{x}^2} + A(\bar{x}) \frac{d\bar{y}}{d\bar{x}} + B(\bar{x}) \bar{y} = C(\bar{x}) \bar{y}^{-3}.$$

Lemma 21.3. The equivalence transformations of equations (3.16) are given by

$$\bar{x} = \varphi(x), \quad \bar{y} = g(x)y \quad (3.29)$$

with two arbitrary functions $\varphi(x) \neq 0$ and $g(x) \neq 0$.

Remark 21.3. The nonlinear equations of the form (3.16) and the linear equations $y'' + a(x)y' + b(x)y =$ have the same equivalence transformations (3.29).

Theorem 21.16. Equation (3.16) can be reduced by an equivalence transformation to the canonical form (3.15) if and only if the condition (3.19) holds, i.e. if equation (3.16) has the form (3.25):

$$y'' + a(x)y' + b(x)y = \frac{k}{y^3} e^{-2 \int a(x) dx}, \quad k = \text{const.} \quad (3.25')$$

The equivalence transformation is given by

$$\bar{x} = \int \frac{e^{-\int a(x) dx}}{f^2(x)} dx, \quad \bar{y} = \frac{y}{f(x)}, \quad (3.30)$$

where $f(x)$ is any non-vanishing solution of the linear equation

$$f'' + a(x)f' + b(x)f = 0. \quad (3.31)$$

Proof. To simplify the calculations, we can write the general equivalence transformation (3.29) in the form

$$\bar{x} = \varphi(x), \quad y = f(x) \bar{y}.$$

Then, denoting $d\bar{y}/d\bar{x} = \bar{y}'$, we have

$$y' = f\varphi' \bar{y}' + f' \bar{y}, \quad y'' = f\varphi'^2 \bar{y}'' + (f\varphi'' + 2f'\varphi') \bar{y}' + f'' \bar{y}.$$

Therefore

$$\begin{aligned} y'' + ay' + by - cy^{-3} \\ = f\varphi'^2 \bar{y}'' + (f\varphi'' + 2f'\varphi' + af\varphi') \bar{y}' + (f'' + af' + bf) \bar{y} - \frac{c}{f^3} \bar{y}^{-3}, \end{aligned}$$

and hence equation (3.16) becomes:

$$\bar{y}'' + \frac{1}{\varphi'} \left(\frac{\varphi''}{\varphi'} + 2\frac{f'}{f}\varphi' + a \right) \bar{y}' + \frac{1}{f\varphi'^2} (f'' + af' + bf) \bar{y} = \frac{c}{f^4\varphi'^2} \bar{y}^{-3} \quad (3.32)$$

In order to arrive at equation (3.15) in the variables \bar{x} and \bar{y} , one has to get rid of the linear terms in \bar{y} and \bar{y}' in (3.32), i.e. to satisfy the equations

$$f'' + af' + bf = 0 \quad (3.33)$$

and

$$\frac{\varphi''}{\varphi'} + 2\frac{f'}{f} + a = 0, \quad (3.34)$$

respectively. Equation (3.33) coincides (3.31), whereas (3.34) yields the first equation (3.30) upon integrating twice:

$$\varphi' = \frac{e^{-\int a(x)dx}}{f^2(x)}, \quad \varphi = \int \frac{e^{-\int a(x)dx}}{f^2(x)} dx. \quad (3.35)$$

Finally, (3.35) yields

$$\frac{c(x)}{f^4 \varphi'^2} = c(x) e^{\int a(x)dx},$$

and hence the right-hand side of equation (3.32) assumes the required form $k \bar{y}^{-3}$ if and only if $c(x) = k e^{-\int a(x)dx}$. This completes the proof.

Remark 21.4. Letting in equation (3.25') $k = 0$ we obtain the classical result that the transformation (3.30) reduces the linear equation $y'' + a(x)y' + b(x)y = 0$ to the simplest form $\bar{y}'' = 0$.

CHAPTER 4

Ordinary differential equations possessing fundamental systems of solutions (following Vessiot–Guldberg–Lie)

Continuing our discussion of problems of integration of ordinary differential equations, let us dwell on the following question: what equations (apart from linear ones) have fundamental systems of solutions, so that the problem of constructing the general solution reduces to finding a finite number of particular solutions? The problem has no direct relation to admitted groups but, as Vessiot [174], Guldberg [58] and Lie [122] showed, it is also solved using group theory.

§ 4.1 The main theorem

We say that a system of ordinary differential equations

$$\frac{dx^i}{dt} = F^i(t, x^1, \dots, x^n), \quad i = 1, \dots, n, \quad (4.1)$$

has a fundamental system of solutions if the general solution of the system can be expressed in terms of a finite number m of arbitrarily chosen particular solutions

$$x_k = (x_k^1, \dots, x_k^n), \quad k = 1, \dots, m, \quad (4.2)$$

by expressions of the form

$$x^i = \varphi^i(x_1, \dots, x_m, C_1, \dots, C_n), \quad (4.3)$$

which contain n arbitrary constants C_1, \dots, C_n . In this case the particular solutions (4.2), which are taken to be functionally independent, are called a *fundamental system of solutions* of (4.1). It is required that the form of the expressions (4.3) be independent of the choice of the specific particular solutions (4.2). However, this does not exclude the possibility that a certain system of equations (4.1) admits several distinct representations (4.3) of the general solution with a different number m of the required particular solutions (see Example 21.18 below). The general form of equations with a fundamental system of solutions was determined by Lie, who proved the following basic theorem (its proof, together with a thorough preliminary discussion and examples, can be found in [122] Ch. 24).

Theorem 21.17. The system of equations (4.1) has a fundamental system of solutions if it can be represented in the form

$$\frac{dx^i}{dt} = T_1(t)\xi_1^i(x) + \dots + T_r(t)\xi_r^i(x) \quad (4.4)$$

such that the operators

$$X_\alpha = \xi_\alpha^i(x) \frac{\partial}{\partial x^i}, \quad \alpha = 1, \dots, r, \quad (4.5)$$

span an r -dimensional Lie algebra. The number m of the necessary particular (fundamental) solutions (4.2) satisfies the condition

$$nm \geq r. \quad (4.6)$$

§ 4.2 Examples

Example 21.13. For a linear homogeneous equation

$$\frac{dx}{dt} = A(t)x$$

we have $n = 1$, $m = 1$, $r = 1$, $X = x \frac{d}{dx}$. The representation (4.3) of the general solution in terms of the particular solution x_1 has the form $x = Cx_1$. Clearly, condition (4.6) is satisfied as an equality.

Example 21.14. For the linear non-homogeneous equation

$$\frac{dx}{dt} = A(t)x + B(t)$$

we have $n = 1$, $m = 2$, $r = 2$, so condition (4.6) is also an equality here. The operators (4.5) are $X_1 = \frac{d}{dx}$, $X_2 = x \frac{d}{dx}$, and their commutator is $[X_1, X_2] = X_1$. Hence they span a two-dimensional Lie algebra L_2 . Furthermore, (4.3) is given by the representation $x = x_1 + C\tilde{x}$ of the general solution x in terms of the particular solution x_1 of the non-homogeneous equation and a particular solution \tilde{x} of the homogeneous equation written as a difference of two solutions of the non-homogeneous equation: $\tilde{x} = x_2 - x_1$.

Example 21.15. An example of a non-linear equation with fundamental solutions is the Riccati equation

$$\frac{dx}{dt} = P(t) + Q(t)x + R(t)x^2. \quad (4.7)$$

It has the special form (4.4) with $r = 3$ and the operators (4.5) given by

$$X_1 = \frac{d}{dx}, \quad X_2 = x \frac{d}{dx}, \quad X_3 = x^2 \frac{d}{dx}, \quad (4.8)$$

which span a Lie algebra L_3 . Formula (4.6) gives $m \geq 3$. Consequently, in order to find the general solution of the Riccati equation with arbitrary coefficients $P(t)$, $Q(t)$ and $R(t)$ one needs at least three particular solutions. In fact it suffices to know three solutions, as any four solutions of the Riccati equation are related by the condition that their cross-ratio be constant. In § 4.3 we shall return to a discussion of the properties of the Riccati equation from the group-theoretical point of view.

Example 21.16. Let us consider a homogeneous system of two linear equations,

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x + a_{12}(t)y, \\ \frac{dy}{dt} &= a_{21}(t)x + a_{22}(t)y. \end{aligned}$$

It has the special form (4.4) with the coefficients

$$\begin{aligned} T_1 &= a_{11}(t), \quad T_2 = a_{12}(t), \quad T_3 = a_{21}(t), \quad T_4 = a_{22}(t), \\ \xi_1 &= (x, 0), \quad \xi_2 = (y, 0), \quad \xi_3 = (0, x), \quad \xi_4 = (0, y). \end{aligned}$$

Therefore in this case the operators (4.5) are

$$X_1 = x \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial y}.$$

One can easily verify that the span a Lie algebra L_4 . The formula (4.3) is provided by the representation of the general solution:

$$x = C_1 x_1 + C_2 x_2, \quad y = C_1 y_1 + C_2 y_2$$

in terms of two particular solutions, (x_1, y_1) and (x_2, y_2) . It follows that $m = 2$ and that the condition (4.6) is an equality since $n = 2$, $r = 4$.

Example 21.17. In the case of a non-homogeneous system

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + b_1(t),$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + b_2(t)$$

one has to add to the coefficients T_α , ξ_α of the previous example the following:

$$T_5 = b_1(t), \quad T_6 = b_2(t), \quad \xi_5 = (1, 0), \quad \xi_6 = (0, 1).$$

Hence, the Lie algebra L_4 of Example 21.16 extends to the algebra L_6 obtained by adding to the basis of L_4 the following two operators:

$$X_5 = \frac{\partial}{\partial x}, \quad X_6 = \frac{\partial}{\partial y}.$$

Thus $n = 2$, $r = 6$, and (4.6) shows that for a representation of the general solution of the above non-homogeneous system with arbitrary coefficients $a_{ij}(t)$ and $b_i(t)$ one has to know three particular solutions (x_k, y_k) , $k = 1, 2, 3$. The corresponding formula (4.3) is well known and has the form:

$$x = x_1 + C_1(x_2 - x_1) + C_2(x_3 - x_1),$$

$$y = y_1 + C_1(y_2 - y_1) + C_2(y_3 - y_1).$$

Example 21.18. In his book [122] Lie gives an example of a non-homogeneous system of two linear equations, where knowledge of two particular solutions is sufficient. Namely, he considers the system

$$\frac{dx}{dt} = a_{12}(t)y + b_1(t),$$

$$\frac{dy}{dt} = -a_{12}(t)x + b_2(t). \quad (4.9)$$

The peculiarity of this system is of group-theoretical nature and consists in the following. By Theorem 21.17, we associate with the system (4.9) the operators

$$X_1 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}.$$

They generate the group of isometric motions (rotations and translations) in the (x, y) -plane. As the motions conserve distances between any two points, any three solutions (x_1, y_1) , (x_2, y_2) , (x, y) of the system (4.9) are connected by two relations:

$$\begin{aligned} (x - x_1)^2 + (y - y_1)^2 &= C_1, \\ (x - x_2)^2 + (y - y_2)^2 &= C_2. \end{aligned} \quad (4.10)$$

Thus, the system (4.9) admits two representations of the general solution: as a linear superposition of three solutions (Example 21.17), and as a “non-linear superposition” of two solutions obtained by solving equalities (4.10) for x, y .

§ 4.3 Projective interpretation of Riccati’s equation

Theorem 21.17 allows one to find all ordinary differential equations (4.1) possessing fundamental systems of solutions by reducing the problem to enumeration of all possible transformation groups with a finite number of parameters (or of the corresponding finite-dimensional Lie groups of operators (4.5)) in the n -dimensional space of variables $x = (x^1, \dots, x^n)$. Such an enumeration has been performed by Lie, e.g. for the cases of the real line ($n = 1$) and of the plane ($n = 2$). These results can also be found in [30], Ch. IV. Here we confine ourselves to the case $n = 1$. Then the choice is severely restricted. Namely, every group of transformations on the real line coincides (up to a change of variables) with the group of projective transformations generated by the three-dimensional Lie group with the generators (4.8), or with one of its subgroups. This means that for $n = 1$ the Riccati equation (4.7) is (up to a change of the variable x) the most general equation with a fundamental system of solutions. For $n \geq 2$ such equations are much more numerous (see [4], [5], [9]).

Thus *the Riccati equation is in a sense a realization of the group of projective transformations*. A reflection of this fact is the theorem that the cross-ratio of any four solutions of the Riccati equation is constant. To prove this theorem, let us introduce the homogeneous coordinates u, v by

setting $x = u/v$. Then the Riccati equation ((4.7')) assumes the form

$$v\left(\frac{du}{dt} - \frac{1}{2}Qu - Pv\right) - u\left(\frac{dv}{dt} + Ru + \frac{1}{2}Qv\right) = 0. \quad (4.7')$$

As the definition $x = u/v$ contains two functions, one can constrain them by one additional relation, e.g. by the equation $\frac{du}{dt} - \frac{1}{2}Qu - Pv = 0$. Then equation (4.7') reduces to the system of two linear equations:

$$\frac{du}{dt} = \frac{1}{2}Q(t)u + P(t)v, \quad \frac{dv}{dt} = -R(t)u - \frac{1}{2}Q(t)v.$$

This is a representation of the Riccati equation in the projective space. Let (u_1, v_1) and (u_2, v_2) be two particular solutions of this system, chosen so that the ratios u_1/u_2 and v_1/v_2 are not equal to the same constant. Then

$$u = C_1u_1 + C_2u_2, \quad v = C_1v_1 + C_2v_2,$$

and hence the general solution to the Riccati equation (4.7) is given by

$$x = \frac{C_1u_1 + C_2u_2}{C_1v_1 + C_2v_2} = \frac{u_1 + Ku_2}{v_1 + Kv_2}. \quad (4.11)$$

Let x_1, \dots, x_4 be four solutions of the Riccati equation corresponding to certain particular values K_1, \dots, K_4 of the constant K in (4.11). Then the following equation holds:

$$\frac{x_1 - x_2}{x_3 - x_2} : \frac{x_1 - x_4}{x_3 - x_4} = \frac{K_1 - K_2}{K_3 - K_2} : \frac{K_1 - K_4}{K_3 - K_4}, \quad (4.12)$$

and hence the cross-ratio of any four solutions of the Riccati equation is constant.

§ 4.4 Linearizable Riccati equations

Thus, by Theorem 21.17, to each Riccati equation (4.7) there corresponds a Lie algebra L_r which is either a three-dimensional algebra with the basis (4.8) or its subalgebra (for certain particular coefficients $P(t)$, $Q(t)$, $R(t)$; in this case $r = 2$ or $r = 1$). This algebra allows one to find out easily whether there is a linearizing change of the variable x for a given Riccati equation. Namely, the following statement is valid [87].

Theorem 21.18. If a Riccati equation (4.7) has one of the following four properties, it also has the other three:

(1) equation (4.7), $\frac{dx}{dt} = P(t) + Q(t)x + R(t)x^2$, is linearizable by a change of the variable x ;

(2) equation (4.7) can be written in the two-term form

$$\frac{dx}{dt} = T_1(t)\xi_1(x) + T_2(t)\xi_2(x) \quad (4.13)$$

so that the operators

$$X_1 = \xi_1(x)\frac{d}{dx}, \quad X_2 = \xi_2(x)\frac{d}{dx} \quad (4.13')$$

generate a two-dimensional Lie algebra, i.e.

$$[X_1, X_2] = \alpha X_1 + \beta X_2.$$

If $[X_1, X_2] = 0$, the algebra is one-dimensional and the Riccati equation in question has separable variables;

(3) equation (4.7) has either the form

$$\frac{dx}{dt} = Q(t)x + R(t)x^2. \quad (4.14)$$

or the form

$$\frac{dx}{dt} = P(t) + Q(t)x + k[Q(t) - kP(t)]x^2 \quad (4.15)$$

with a constant (in general complex) coefficient k ;

(4) equation (4.7) has a constant (in general complex) solution.

Proof. Let us demonstrate that, e.g. the properties (1) and (2) are equivalent. A key factor in the proof is that equation (4.13) and the Lie algebra structure $[X_1, X_2] = \alpha X_1 + \beta X_2$ remain unaltered under any change of the dependent variable x .

(1) \Rightarrow (2). If equation (4.7) is linearizable, we can assume that it is already written in the linear form and the property (2) follows from Examples 21.13 and 21.14 in § 4.2.

(2) \Rightarrow (1). Let the operators (4.13') span a Lie algebra L_r of dimension $r \leq 2$. The case $r = 1$ yields $\xi_2(x) = c\xi_1(x)$, $c = \text{const}$. Then (4.13) is an equation with separable variables:

$$\frac{dx}{dt} = [T_1(t) + cT_2(t)]\xi_1(x).$$

It becomes linear after introducing a canonical variable for $X_1 = \xi_1(x)\frac{d}{dx}$. Indeed, defining the canonical variable y by the equation

$$X_1(y) \equiv \xi_1(x)\frac{dy}{dx} = 1,$$

one can rewrite the above equation in the linear form:

$$\frac{dy}{dt} = T_1(t) + cT_2(t).$$

Suppose now that $r = 2$. In order to linearize equation (4.13) we will transform the operators (4.13') to the form (cf. Examples 21.13 and 21.14)

$$X_1 = \frac{d}{dx}, \quad X_2 = x \frac{d}{dx}. \quad (4.13'')$$

One can assume that the first operator (4.13') is already written in the canonical variable x , i.e. it has the form $X_1 = \frac{d}{dx}$ given in (4.13''). Let

$$X_2 = f(x) \frac{d}{dx}$$

be the second operator (4.13'). We have

$$[X_1, X_2] = f'(x) \frac{d}{dx}$$

and the equation $[X_1, X_2] = \alpha X_1 + \beta X_2$ is written

$$\frac{df}{dx} = a + bf.$$

It follows that at least one of the coefficients a or b is not zero since the operators X_1 and X_2 are linearly independent, and hence $f'(x) \neq 0$. Integrating the above equation we obtain:

$$f = ax + C \implies X_2 = ax \frac{d}{dx} + CX_1, \quad \text{if } b = 0,$$

$$f = Ce^{bx} - \frac{a}{b} \implies X_2 = Ce^{bx} \frac{d}{dx} - \frac{a}{b} X_1, \quad \text{if } b \neq 0.$$

In the first case, a basis of L_2 has already the form (4.13''). In the second case, one can assign bx as new x and take a basis of L_2 in the form

$$X_1 = \frac{d}{dx}, \quad X_2 = e^x \frac{d}{dx}$$

and transform them to the form (4.13'') by the substitution $\bar{x} = e^{-x}$.

It remains to show that the equation $[X_1, X_2] = 0$ implies that the operators (4.13') are linearly dependent. Indeed, the equation $[X_1, X_2] = 0$ is written $\xi_1 \xi_2' - \xi_2 \xi_1' = 0$, or

$$\frac{d\xi_2}{\xi_2} = \frac{d\xi_1}{\xi_1},$$

and the integration yields $\xi_2(x) = \alpha \xi_1(x)$, $\alpha = \text{const.}$

Remark 21.5. Equation (4.15) has a constant solution $x = -1/k$. Therefore the linear equation, which is a particular case of (4.15) for $k = 0$, can be considered as a Riccati equation, which has the point at infinity as its constant solution.

CHAPTER 5

The invariance principle in problems of mathematical physics

Special types of exact solutions, widely known now as *invariant solutions*, have long been successfully applied to the analysis of specific problems and were in accepted use in mechanics and physics prior to the development of group theory, becoming in the process a sort of folklore. Lie uncovered the group-theoretical meaning of these solutions and considered the possibility of using invariant solutions for integrating partial differential equations with a sufficiently large group of symmetries ([123], Chapters III and IV). Subsequently group theory made it possible to understand, clarify, and extend many intuitive ideas, as a result of which the method of invariant solutions could be included as an important integral part of contemporary group analysis. It was precisely through the concept of invariant solutions that the theater of action of group analysis shifted in the 1960s from ordinary differential equations to problems of mechanics and mathematical physics (largely due to the books [24], [159], [148], [138], [2]-[3]).

In the passage from ordinary to partial differential equations it becomes impossible (barring rare exceptional cases), as well as practically futile, to write down a general solution of the equation. Instead, mathematical physics deals with solutions that satisfy *a priori* fixed side (initial, boundary, and so on) conditions. In the solution of many problems of mathematical physics the following semi-empirical principle (it has been rigorously justified so far only in limited cases; see [144], §29, [24], §89, [130], [152]) may be used with success.

Invariance principle: *If a boundary-value problem is invariant with respect to a group, then one should look for a solution in the class of functions invariant with respect to that group.*

Invariance of a boundary-value problem presupposes the invariance of the differential equation and of the manifold containing the data, as well as the data given on that manifold. If the boundary conditions violate invariance (which happens frequently), the invariance principle is useful in conjunction with other methods. For instance, it will be shown below in

§ 5.3 that Riemann's method reduces the Cauchy problem with arbitrary (and therefore not invariant) data to a particular problem (Goursat's problem) which turns out to be invariant, and hence can be solved by applying the invariance principle.

§ 5.1 Application of the invariance principle to a nonlinear boundary value problem

Consider the following nonlinear equation

$$\Delta u = e^u, \quad (5.1)$$

where $u = u(x, y)$ and $\Delta u = u_{xx} + u_{yy}$ is the Laplacian with two independent variables. Equation (5.1) admits the operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} - 2\xi_x \frac{\partial}{\partial u}, \quad (5.2)$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary solutions of the Cauchy-Riemann system

$$\xi_x - \eta_y = 0, \quad \xi_y + \eta_x = 0. \quad (5.3)$$

Consequently, one can express the general solution of the nonlinear equation (5.1) via the solution of the Laplace equation

$$\Delta v = 0 \quad (5.4)$$

in the form

$$u = \ln \left(2 \frac{v_x^2 + v_y^2}{v^2} \right). \quad (5.5)$$

In other words, the nonlinear equation (5.1) is mapped to the linear equation (5.4) by the transformation (5.5). However, this transformation is not particularly useful in dealing with concrete problems. Let us turn to an example.

Consider the following boundary value problem in the circle of radius $r = 1$:

$$\Delta u = e^u, \quad u|_{r=1} = 0, \quad (5.6)$$

where $r = \sqrt{x^2 + y^2}$. The general solution (5.5) is not suitable for solving our problem since it leads to the nonlinear boundary-value problem

$$\Delta v = 0, \quad \left(v_x^2 + v_y^2 - \frac{1}{2}v^2 \right) \Big|_{r=1} = 0.$$

In order to solve the problem (5.6) it is convenient to use the polar coordinates

$$x = r \cos \varphi, \quad y = r \sin \varphi. \quad (5.7)$$

In these coordinates Equation (5.1) is written

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = e^u. \quad (5.8)$$

Since the differential equation and the boundary condition in the problem (5.6) are invariant with respect to the rotation group, we can apply the invariance principle and seek the solution depending only on the invariant variable r . Then Equation (5.8) is written

$$u_{rr} + \frac{1}{r}u_r = e^u. \quad (5.9)$$

We will assume that the function $u(r)$ is bounded at the “singular” point $r = 0$ and formulate the boundary conditions of the problem (5.6) in the following form:

$$u(1) = 0, \quad u(0) < \infty. \quad (5.10)$$

One can integrate equation (5.9) by means of Lie’s method. Namely, it has two infinitesimal symmetries

$$X_1 = r \frac{\partial}{\partial r} - 2 \frac{\partial}{\partial u}, \quad X_2 = r \ln r \frac{\partial}{\partial r} - 2(1 + \ln r) \frac{\partial}{\partial u}. \quad (5.11)$$

We reduce the first operator to the form $X_1 = \frac{\partial}{\partial t}$ by the change of variables

$$t = \ln r, \quad z = u + 2 \ln r.$$

Equation (5.9) is written in these variables as follows

$$\frac{d^2 z}{dt^2} = e^z. \quad (5.12)$$

Integration by means of the standard substitution $\frac{dz}{dt} = p(v)$ yields

$$\int \frac{dz}{\sqrt{C_1 + 2e^z}} = t + C_2. \quad (5.13)$$

Evaluating the integral in (5.13) one can verify that the condition $u(0) < \infty$ is not satisfied if $C_1 \leq 0$. Therefore we calculate the integral for $C_1 > 0$. For the sake of convenience we set $C_1 = \lambda^2$ and $C_2 = \ln C$. Then we evaluate

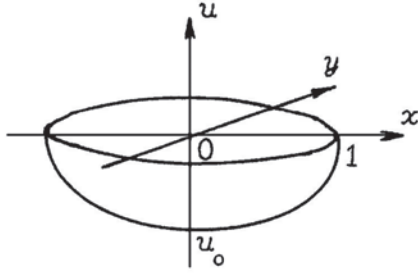


Figure 5: Solution (5.15) with $c^2 = 5 - 2\sqrt{6}$ is bounded.

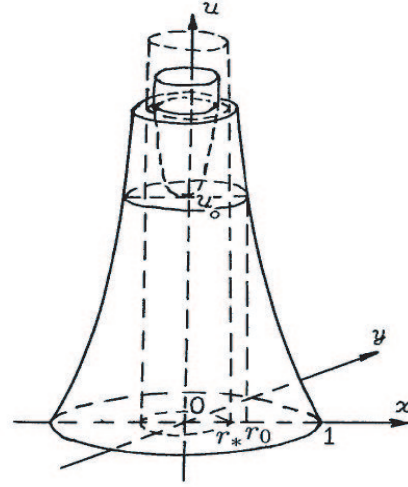


Figure 6: Unbounded solution.

the integral, rewrite the result in the old variables and obtain the following solution:

$$u = \ln \frac{2\lambda^2 (cr)^\lambda}{r^2 [1 - (cr)^\lambda]^2}. \quad (5.14)$$

It follows that

$$u \approx (\lambda - 2) \ln r \quad (r \rightarrow 0).$$

Therefore the condition $u(0) < \infty$ entails that $\lambda = 2$. Furthermore, the boundary condition $u(1) = 0$ takes the form $8c^2 = (1 - c^2)^2$ whence $c^2 = 5 \pm 2\sqrt{6}$.

Hence, the problem (5.6) has two solutions given by the equation

$$u = \ln(8c^2) - \ln(1 - c^2 r^2)^2 \quad (5.15)$$

with

$$c^2 = 5 - 2\sqrt{6}$$

and with

$$c^2 = 5 + 2\sqrt{6},$$

respectively.

The first solution (Fig. 5), i.e. (5.15) corresponding to the case

$$c^2 = 5 - 2\sqrt{6}$$

is bounded everywhere in the circle

$$x^2 + y^2 \leq 1.$$

The second solution (Fig. 6) corresponding to the case

$$c^2 = 5 + 2\sqrt{6}$$

is unbounded on the circle

$$x^2 + y^2 = r_*^2, \quad \text{where} \quad r_* = 1/c \approx 0.33.$$

In this solution, we have $u(0) = u(r_0) = u_0 \approx 4.37$, where $r_0 = \sqrt{2}/c$.

§ 5.2 Spherical functions

Let us consider Laplace's equation $\Delta u = 0$ in three independent variables x, y, z , and let us find its solutions that are invariant with respect to rotations and stretchings in the independent variables and the function u .

It is convenient to use the spherical coordinates

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta$$

($0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$) and write Laplace's operator in the form

$$\Delta = \frac{1}{r^2} \left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \Delta_2 \right),$$

where Δ_2 is the *Laplace–Beltrami operator* on the unit sphere defined by

$$\Delta_2 u = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}.$$

The solutions that are invariant with respect to spatial rotations (spherically symmetric solutions) have the form $u = R(r)$. For the Laplace's equation yield $(r^2 R')' = 0$, whence

$$R = \frac{C_1}{r} + C_2. \quad (5.16)$$

For the group of stretchings in u with the symbol $X = u \frac{\partial}{\partial u}$ any invariant is a function of x, y, z . Therefore no solution $u = \varphi(x, y, z)$ can be invariant with respect to this group, as u is not an invariant.

Let us find invariant solutions for the group of stretchings in the independent variables and the function u :

$$\bar{x} = ax, \quad \bar{y} = ay, \quad \bar{z} = az, \quad \bar{u} = a^n u. \quad (5.17)$$

A function $u = \varphi(x, y, z)$ that is invariant with respect to the stretchings (5.17) is called *homogeneous of degree n* . Thus, let us consider solutions of

Laplace's equation that are homogeneous of degree n . Keeping in mind that the homogeneous solutions will be used for an expansion of an arbitrary solution, we assume that n is an integer.

Writing the operator of the group of stretchings (5.17) in the spherical coordinates in the form

$$X = r \frac{\partial}{\partial r} + nu \frac{\partial}{\partial u}$$

and solving the equation $XJ = 0$ one can see that a basis of invariants is furnished by $\lambda = ur^{-n}$, θ , φ . Hence, the invariant equation is written $\lambda = Y_n(\theta, \varphi)$ (the index n means that for each choice of n one has a different function Y) and provides the following general form of the invariant solution:

$$u = r^n Y_n(\theta, \varphi). \quad (5.18)$$

Then

$$\Delta u = \frac{1}{r^2} \left[\frac{\partial}{\partial r} (nr^{n+1} Y_n) + r^n \Delta_2 Y_n \right] = r^{n-2} [n(n+1) Y_n + \Delta_2 Y_n],$$

and Laplace's equation assumes the form

$$\Delta_2 Y_n + n(n+1) Y_n = 0. \quad (5.19)$$

Thus, the coefficient Y_n of the invariant solution (5.18) is an eigenfunction of the Laplace–Beltrami operator Δ_2 with the eigenvalue $n(n+1)$. It is known as the *spherical function of order n* .

Equation (5.19) is invariant under the change $n = -(m+1)$, since $n(n+1) = m(m+1)$. Therefore, if (5.18) is a solution of Laplace's equation, the following function is also a solution:

$$r^{-(n+1)} Y_n(\theta, \varphi). \quad (5.18')$$

The series composed of the invariant solutions (5.16), (5.18) and (5.18') with constant coefficients,

$$u(r, \theta, \varphi) = \left(A_0 + \frac{B_0}{r} \right) + \sum_{n=1}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) Y_n(\theta, \varphi), \quad (5.20)$$

is a (formal) solution of Laplace's equation in view of the superposition principle (the latter being also a group property). The formula (5.20) gives a representation of the general solution and is usually derived by the method of separation of variables (see, e.g. [172]).

§ 5.3 A group-theoretical touch to Riemann's method

This section is of a synthetic nature, being a result of combining Riemann's method [153] for integrating second-order linear hyperbolic equations with

Lie's classification [112] of such equations. One can find in [34] a detailed description of known methods of constructing Riemann's function (called in [34] the Riemann–Green function) for particular types of equations. Specifically, six methods are described there. The group-theoretical approach presented below provides the seventh method.

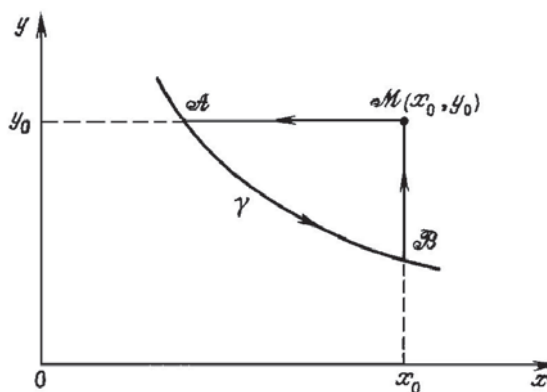


Figure 7: Riemann's method.

Riemann's method reduces the problem of integration of the equation

$$L[u] \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y) \quad (5.21)$$

to that of constructing the auxiliary function v solving the adjoint equation

$$L^*[v] \equiv v_{xy} - (av)_x - (bv)_y + cv = 0 \quad (5.22)$$

and satisfying the following conditions on the characteristics:

$$v|_{x=x_0} = e^{\int_{y_0}^y a(x_0, \eta) d\eta}, \quad v|_{y=y_0} = e^{\int_{x_0}^x b(\xi, y_0) d\xi}. \quad (5.23)$$

Provided that the function v is known, the solution of the Cauchy problem

$$L[u] = f, \quad u|_{\gamma} = u_0(x), \quad u_y|_{\gamma} = u_1(x) \quad (5.24)$$

with data on an arbitrary non-characteristic curve γ is given by the formula

$$u(x_0, y_0) = \frac{1}{2}[u(A)v(A) + u(B)v(B)] + \int_{AB} \left\{ \left[\frac{1}{2}vu_x + \left(bv - \frac{1}{2}v_x \right)u \right] dx - \left[\frac{1}{2}vu_y + \left(av - \frac{1}{2}v_y \right)u \right] dy \right\} + \iint v f dx dy,$$

where the double integral is taken over the domain bounded by the characteristics $x = x_0$, $y = y_0$ and the curve γ (see Fig. 7). The function v is called *Riemann's function*, and the boundary-value problem (5.22), (5.23) is called the *characteristic Cauchy problem*, or the *Goursat problem*. The solution of the Goursat problem is unique.

Example 21.19. For the wave equation $u_{xy} = 0$ the corresponding Goursat problem has the simple form

$$v_{xy} = 0, \quad v|_{x=x_0} = 1, \quad v|_{y=y_0} = 1.$$

Clearly its solution is the function $v = 1$.

Example 21.20. The telegraph equation

$$u_{xy} + u = 0 \tag{5.25}$$

is one of the simplest equations (after $u_{xy} = 0$) to which Riemann's method is applicable. In this case, the Goursat problem (5.22), (5.23) has the form

$$v_{xy} + v = 0, \quad v|_{x=x_0} = 1, \quad v|_{y=y_0} = 1. \tag{5.26}$$

Usually, textbooks offer the following “method” for solving it: let us look for a solution of the problem (5.26) in the form

$$v = V(z), \quad z = (x - x_0)(y - y_0). \tag{5.27}$$

This leads to the ordinary differential equation

$$zV'' + V' + V = 0,$$

which is Bessel's equation and assumes the standard form

$$\mu V'' + V' + \mu V = 0$$

upon the substitution $\mu = \sqrt{4z}$. Thus the Riemann function for the telegraph equation (5.25) is expressed in terms of Bessel's function J_0 in the form

$$v(x, y; x_0, y_0) = J_0(\sqrt{4(x - x_0)(y - y_0)}). \tag{5.28}$$

Example 21.21. Riemann himself applied the method he suggested to the following equation (see [153], §9):

$$v_{xy} + \frac{l}{(x + y)^2}v = 0, \quad l = \text{const}. \tag{5.29}$$

For this equation the conditions (5.23) on the characteristics are written

$$v|_{x=x_0} = 1, \quad v|_{y=y_0} = 1. \tag{5.30}$$

Riemann reduces the resulting characteristic Cauchy problem to an ordinary differential equation (which defines a special Gauss' hypergeometric function) by assuming that v is a function of the one variable

$$z = \frac{(x - x_0)(y - y_0)}{(x_0 + y_0)(x + y)}. \tag{5.31}$$

In the last two examples we could reduce the construction of the Riemann function to the solution of an ordinary differential equation by having guessed correctly the form of the solution of the Goursat problem. It turns out that the ability to perform such a reduction is related to the fact that the equation admits a sufficiently large group, and to the invariance of the conditions on the characteristics with respect to a one-parameter subgroup of that group. It is the invariance principle, and not a lucky guess, that gives us the correct form of the solution. Let us consider the equations of Examples 21.20 and 21.21 from this point of view.

Example 21.22. The telegraph equation (5.25) admits the three-parameter group (in addition to stretchings of the variable u and the infinite group consisting of addition to u of any solution of the equation; this group is common to all linear equations) with the generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

Let us find a linear combination of these operators,

$$X = \alpha X_1 + \beta X_2 + \gamma X_3,$$

admitted by the Goursat problem (5.26). Let us require first the invariance of the characteristics $x = x_0$, $y = y_0$. The invariance test has the form

$$[X(x - x_0)]_{x=x_0} \equiv \alpha + \gamma x_0 = 0, \quad [X(y - y_0)]_{y=y_0} \equiv \beta - \gamma y_0 = 0.$$

It follows that $\gamma \neq 0$, since otherwise $\alpha = \beta = 0$. Therefore one can set $\gamma = 1$ and obtain $\alpha = -x_0$, $\beta = y_0$. One can readily verify that the resulting operator

$$X = (x - x_0) \frac{\partial}{\partial x} - (y - y_0) \frac{\partial}{\partial y} \quad (5.32)$$

is admitted by the Goursat problem (5.26). Therefore one can use the invariance principle and look for the solution to the Goursat problem among invariant functions with respect to the one-parameter group with the generator (5.32). This group has two independent invariants, namely v and $z = (x - x_0)(y - y_0)$. Therefore the invariant solution has the form (5.27).

Example 21.23. Equation (5.29) also admits three operators, namely:

$$X_1 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}.$$

As in the previous example, one can find uniquely their linear combination,

$$X = (x - x_0)(x + y_0)\frac{\partial}{\partial x} - (y - y_0)(y + x_0)\frac{\partial}{\partial y}, \quad (5.33)$$

leaving invariant the characteristics and the conditions on the characteristics. Therefore we shall look for the solution of the problem (5.29), (5.30) in the class of invariant functions. Since the invariants for the operator (5.33) are v and

$$\mu = \frac{(x - x_0)(y - y_0)}{(x + y_0)(y + x_0)}, \quad (5.34)$$

the invariant solution has the form $v = v(\mu)$. This is the invariant solution found by Riemann; the variable z (5.31) he used is related to the invariant μ (5.34) by the functional relation $z = \mu/(1 - \mu)$, and hence is also an invariant.

Let us now move on to general statements. The starting point here will be Lie's result [112] on the group classification of homogeneous equations (5.21). Most convenient for our aims is the invariant (with respect to equivalence transformations of linear equations) formulation of this result due to Ovsyannikov [136] (see also [144], §9.6). Note that in the assertions below the admitted group is considered modulo addition to the dependent variable of an arbitrary solution $\varphi(x, y)$ of the linear equation. In other words, by an admitted algebra we mean the quotient algebra of the general admitted algebra by the ideal spanned by the operators $X = \varphi(x, y)\frac{\partial}{\partial u}$. So let us start with the following result of group classification.

Theorem 21.19. Let us denote by h and k the Laplace invariants of Eq. (5.36):

$$h = a_x + ab - c, \quad k = b_y + ab - c. \quad (5.35)$$

The equation

$$L[u] \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0 \quad (5.36)$$

admits a four-dimensional Lie algebra if and only if the quantities*

$$p = \frac{k}{h}, \quad q = \frac{1}{h} \frac{\partial^2 \ln |h|}{\partial x \partial y} \quad (5.37)$$

* *Author's note to this 2006 edition:* It is assumed that $h \neq 0$. If $h = 0, k \neq 0$ then h and k have to be interchanged. If $h = k = 0$ equation (5.36) is equivalent to $u_{xy} = 0$.

are constant. It admits at most a two-dimensional algebra if at least one of the quantities (5.37) is non-constant. Furthermore, in the case of constant p and q equation (5.36) reduces either to the form (if $q = 0$)

$$u_{xy} + xu_x + pyu_y + pxyu = 0 \quad (5.38)$$

and admits the operator

$$X = (\alpha_1 x + \alpha_2) \frac{\partial}{\partial x} + (-\alpha_1 y + \alpha_3) \frac{\partial}{\partial y} - (\alpha_2 y + \alpha_3 p x + \alpha_4) u \frac{\partial}{\partial u},$$

or to the form (if $q \neq 0$)

$$u_{xy} - \frac{2}{q(x+y)} u_x - \frac{2p}{q(x+y)} u_y + \frac{4p}{q^2(x+y)^2} u = 0 \quad (5.39)$$

and admits the operator

$$X = (\alpha_1 x^2 + \alpha_2 x + \alpha_3) \frac{\partial}{\partial x} + (-\alpha_1 y^2 + \alpha_2 x - \alpha_3) \frac{\partial}{\partial y} + \left[\alpha_1 \frac{2}{q} (px - y) + \alpha_4 \right] u \frac{\partial}{\partial u}.$$

Corollary 21.2. The adjoint equation (5.22) admits a four-dimensional algebra if and only if so does the original equation (5.36).

Indeed, in the mutually adjoint equations $L[u] = 0$ and $L^*[u] = 0$ the Laplace invariants h , k and h_* , k_* are related by the permutation, $h = k_*$, $k = h_*$. This follows immediately from (5.22), (5.35). Therefore the corresponding invariants (5.37) p , q and p_* , q_* are either constant for both equations, or non-constant.

Remark 21.6. The general equivalence transformation of equations (5.36) has the form

$$\bar{x} = f(x), \quad \bar{y} = g(y), \quad \bar{u} = \lambda(x, y)u. \quad (5.40)$$

The Laplace invariants (5.35) are in reality invariant only with respect to the linear transformation of the dependent variable, $\bar{u} = \lambda(x, y)u$, without a change of the coordinates x, y . On the other hand, the quantities p and q defined by (5.37) are invariant with respect to the general equivalence transformation (5.40) (see [136]) and therefore arise naturally in the statement of Theorem 21.19.

Now we can prove the main result of this section.

Theorem 21.20. [89]. Let equation (5.21) have constant invariants (5.37). Then the Goursat problem (5.22), (5.23) admits a one-parameter group and Riemann's function is determined by a second-order ordinary differential equation.

Proof. By Theorem 21.19 and its corollary, it suffices to consider equations (5.21) for which the adjoint equation (5.22) has already been reduced by a suitable equivalence transformation either to the form (5.38) or to the form (5.39), depending on whether the invariant q_* of the conjugate equation vanishes or not.

Let us consider the case $q_* = 0$ and assume that the adjoint equation has been reduced to the form (5.38):

$$L^*[u] \equiv v_{xy} + xv_x + p_*yv_y + p_*xv = 0, \quad p_* = \text{const.} \quad (5.41)$$

Then the conditions (5.23) have the form

$$v|_{x=x_0} = e^{x_0(y_0-y)}, \quad v|_{y=y_0} = e^{p_*y_0(x_0-x)} \quad (5.42)$$

and the Goursat problem (5.41), (5.42) admits the operator

$$X = (x - x_0)\frac{\partial}{\partial x} - (y - y_0)\frac{\partial}{\partial y} + [x_0(y - y_0) - p_*y_0(x - x_0)]v\frac{\partial}{\partial v}.$$

This operator has the invariants

$$V = ve^{[x_0(y-y_0)+p_*y_0(x-x_0)]} \quad \text{and} \quad \mu = (x - x_0)(y - y_0).$$

By the invariance principle, the solution of the Goursat problem (5.41), (5.42) is sought in the invariant form

$$v = e^{[x_0(y_0-y)+p_*y_0(x_0-x)]}V(\mu), \quad \mu = (x - x_0)(y - y_0).$$

By substituting into (5.41) we obtain the ordinary differential equation

$$\mu V'' + [1 + (p_* + 1)\mu]V' + p_*\mu V = 0 \quad (5.43)$$

for the function $V(\mu)$. Two conditions (5.42) on the characteristics yield the single initial condition $V(0) = 1$. So the construction of the Riemann function requires the integration of equation (5.43) with this initial condition.

Suppose now that $q_* \neq 0$ and that the adjoint equation has been reduced to the form (5.39):

$$L_*[v] \equiv v_{xy} - \frac{2}{q_*(x+y)}v_x - \frac{2p_*}{q_*(x+y)}v_y + \frac{4p_*v}{q_*^2(x+y)^2} = 0, \quad (5.44)$$

where $p_*, q_* = \text{const.}$ In this case the conditions (5.23) are of the form

$$v|_{x=x_0} = \left(\frac{x_0+y}{x_0+y_0}\right)^{2/q_*}, \quad v|_{y=y_0} = \left(\frac{x+y_0}{x_0+y_0}\right)^{2p_*/q_*}, \quad (5.45)$$

and the Goursat problem (5.44), (5.45) admits the operator

$$X = (x - x_0)(x + y_0) \frac{\partial}{\partial x} - (y - y_0)(x_0 + y) \frac{\partial}{\partial y} + \frac{2}{q_*} [p_*(x - x_0) - (y - y_0)] v \frac{\partial}{\partial v}.$$

The invariants are

$$V = (x + y_0)^{-2p_*/q_*} (x_0 + y)^{-2/q_*} v, \quad \mu = \frac{(x - x_0)(y - y_0)}{(x + y_0)(x_0 + y)},$$

and an invariant solution is written in the form

$$v = \left(\frac{x + y_0}{x_0 + y_0} \right)^{2p_*/q_*} \left(\frac{x_0 + y}{x_0 + y_0} \right)^{2/q_*} v(\mu), \quad \mu = \frac{(x - x_0)(y - y_0)}{(x + y_0)(x_0 + y)}.$$

As a result, the Goursat problem (5.44), (5.45) is reduced to the solution of the ordinary differential equation

$$\mu(1 - \mu)^2 V'' + (1 - \mu) \left[1 + \left(\frac{2(p_* + 1)}{q_*} - 1 \right) \mu \right] V' + \frac{4p_*}{q_*^2} \mu V = 0 \quad (5.46)$$

under the condition $V(0) = 1$. This completes the proof.

Example 21.24. Let us consider the equation

$$u_{xy} + \frac{l}{x + y} u = 0, \quad l = \text{const.} \neq 0,$$

which holds an “intermediate” position between equations (5.25) and (5.29). The Laplace invariants (5.35) are $h = k = -l/(x + y)$, and the formulae (5.37) yield $p = 1$, $q = 1/[l(x + y)]$. As q is not constant, Theorem 21.20 cannot be applied.

For practical use of Theorem 21.20 in the case of constant p and q , there is no need to reduce the adjoint equation to the corresponding standard form (5.41) or (5.44). It can turn out that the computations are easier in the original variables. For that we only need to find the admitted operators. The corresponding algorithm is described in Table 6.10, and it will be easy to remember after the following additional clarifications.

If for an equation (5.36) one is looking for an admitted operator of the form

$$X = \xi(x, y, u) \frac{\partial}{\partial x} + \eta(x, y, u) \frac{\partial}{\partial y} + \mu(x, y, u) \frac{\partial}{\partial u},$$

then, after a preliminary analysis of the determining equations, one can verify that this operator has the following special form:

$$X = \xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} + \sigma(x, y) u \frac{\partial}{\partial u}. \quad (5.47)$$

Hence, the group admitted by equation (5.36) belongs in the class of equivalence transformations (5.40). This is the first useful observation.

Subsequent analysis of the determining equations shows ([144], §9.4) that the operator (5.47) is admitted by (5.36) if and only if the following equations hold:

$$\begin{aligned} (h\xi)_x + (h\eta)_y &= 0, & (k\xi)_x + (k\eta)_y &= 0, \\ (\sigma + a\eta + b\xi)_x &= (h - k)\eta, & (\sigma + a\eta + b\xi)_y &= (k - h)\xi. \end{aligned} \quad (5.48)$$

The second useful observation concerns these equations and allows one to give them a clear geometric meaning.

Let us start by noting that up till now we have actively used the tangent vector field of a one-parameter group in the form of a *first-order linear differential operator* (1.7). If we now take into account the close connection between linear partial differential equations and equations in differentials (equations of characteristics), then it becomes natural to use along with the generator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$$

its *dual differential form*

$$\Xi = \eta dx - \xi dy. \quad (5.49)$$

Furthermore, the differential form (5.49) will be considered to be dual with respect to the operator (5.47) as well. In this way we emphasize the fact that transformations of x, y have a geometrical meaning, whereas the transformation rule for the dependent variable u is defined by its physical significance, and not by geometry.

Using the differential form (5.49), the determining equations (5.48) are written

$$d(h\Xi) = 0, \quad d(k\Xi) = 0, \quad d(\sigma + a\eta + b\xi) = (h - k)\Xi. \quad (5.48')$$

Since every closed form is locally exact, the first two equations (5.48) mean that both Laplace invariants h and k (5.35) are integrating factors for the dual differential form (5.49) to the operator (5.47). In view of this, in particular, the form $(h - k)\Xi$ is exact. Therefore the last equation in (5.48') can be integrated and gives us the coefficient σ of the operator (5.47).

Example 21.25. Consider the equation

$$u_{xy} - \frac{l}{x+y}(u_x + u_y) = f(x, y), \quad l \neq 0, \quad l \neq -1. \quad (5.50)$$

Here $p = 1$, $q = 2/[l(l + 1)]$ and Theorem 21.20 is applicable.

The adjoint equation is

$$v_{xy} + \frac{l}{x+y} (v_x + v_y) - \frac{2l}{(x+y)^2} v = 0 \quad (5.51)$$

and the conditions on the characteristics have the form

$$v|_{x=x_0} = \left(\frac{x_0 + y_0}{y + x_0} \right)^l, \quad v|_{y=y_0} = \left(\frac{x_0 + y_0}{x + y_0} \right)^l. \quad (5.52)$$

Let us find the algebra admitted by equation (5.51). The Laplace invariants for equation (5.51) are equal and coincide with the invariants $h = k$ of equation (5.50) (see Corollary of Theorem 21.19). Therefore the first two equations (5.48) reduce to the single equation

$$\left(\frac{\eta}{(x+y)^2} \right)_y + \left(\frac{\xi}{(x+y)^2} \right)_x = 0,$$

whence

$$\xi = C_1 x^2 + C_2 x + C_3, \quad \eta = -C_1 y^2 + C_2 y - C_3.$$

Now integration of the third equation in (5.48'), which has the form

$$d\left(\sigma + \frac{l}{x+y}(\xi + \eta)\right) = 0,$$

yields $\sigma = C_4 + C_1 l(y-x)$. Hence, the operator admitted by equation (5.51) is has the form

$$X = (C_1 x^2 + C_2 x + C_3) \frac{\partial}{\partial x} - (C_1 x^2 - C_2 y + C_3) \frac{\partial}{\partial y} + [C_4 + C_1 l(y-x)] v \frac{\partial}{\partial v}$$

and contains four arbitrary constants C_1, \dots, C_4 , in accordance with Theorem 21.19. This operator, upon using the invariance condition of equations (5.52) on the characteristics, becomes

$$X = (x-x_0)(x+y_0) \frac{\partial}{\partial x} - (y-y_0)(y+x_0) \frac{\partial}{\partial y} + l[(y-y_0) - (x-x_0)] v \frac{\partial}{\partial v}. \quad (5.53)$$

Thus, the Goursat problem (5.51), (5.52) admits the one-parameter group with the symbol (5.53). The invariant solution is written

$$v = \frac{(x_0 + y_0)^{2l}}{(x + y_0)^l (y + x_0)^l} V(z), \quad z = \frac{(x - x_0)(y - y_0)}{(x_0 + y_0)(x + y)}.$$

Substitution into (5.51) yields the ordinary differential equation

$$z(1+z)V'' + (1+2(1-l)z)V' - l\left(2 + \frac{l}{1+z}\right)V = 0.$$

One can easily verify that two conditions (5.52) on the characteristics lead to the single initial condition $V(0) = 1$.

§ 5.4 Symmetry of fundamental solutions, or the first steps in group analysis in the space of distributions

Continuing in the direction of the previous section, namely, application of the invariance principle to boundary-value problems with arbitrary data by reduction to an invariant problem of a special type, let us consider the potential of group theory in the construction of fundamental solutions for the three classical equations of mathematical physics. This natural path of development of group analysis venturing into the space of distributions (generalized functions) has been sketched in the booklet [87], where one can find some additional heuristic considerations which are omitted here. On the other hand, here we use infinitesimal techniques to a greater extent than in [87].

§ 5.4.1 Something about distributions

We will consider the general n -dimensional case, so *test functions* $\varphi(x)$ are infinitely differentiable functions of $x = (x^1, \dots, x^n)$ with compact support. The action of a distribution f on $\varphi(x)$ is denoted by $\langle f, \varphi \rangle$ or by $\langle f(x), \varphi(x) \rangle$ when different variables are in use.

Transformation of distributions. Consider a given transformation in \mathbb{R}^n ,

$$\bar{x} = \Phi(x), \quad (5.54)$$

where Φ is a C^∞ diffeomorphism. The Jacobian of this transformation is denoted by $J = \det(\frac{\partial \bar{x}^i}{\partial x^j})$. Let us define a *transformation of distributions* by the equation

$$\langle f(x), \varphi(x) \rangle = \langle (\bar{f} \circ \Phi^{-1})(\bar{x}), (\varphi \circ \Phi^{-1})(\bar{x}) \rangle \quad (5.55)$$

expressing the invariance condition for a functional.

Let us start, as usual, with the regular case when f is locally integrable and

$$\langle f(x), \varphi(x) \rangle = \int_{\mathbb{R}^n} f(x) \varphi(x) dx.$$

The formula for a change of variables in the integral yields

$$\int_{\mathbb{R}^n} f(x) \varphi(x) dx = \int_{\mathbb{R}^n} (f \circ \Phi^{-1})(\bar{x}) (\varphi \circ \Phi^{-1})(\bar{x}) |J|^{-1} d\bar{x},$$

that is (see [31], p. 456)

$$\langle f(x), \varphi(x) \rangle = \langle |J|^{-1} (f \circ \Phi^{-1})(\bar{x}), (\varphi \circ \Phi^{-1})(\bar{x}) \rangle. \quad (5.56)$$

Comparison of (5.55) and (5.56) leads to the following formula for transforming arbitrary distributions:

$$\bar{f} = |J|^{-1} f \quad \text{or} \quad f = \left| \det \left(\frac{\partial \bar{x}^i}{\partial x^j} \right) \right| \bar{f}. \quad (5.57)$$

Taking, in particular, $f = \delta$, we obtain from (5.57) the following transformation of the δ -function:

$$\delta = \left| \det \left(\frac{\partial \bar{x}^i}{\partial x^j} \right) \right|_{x=0} \bar{\delta}. \quad (5.58)$$

Consider now a one-parameter transformation group. Taking, instead of (5.54), the infinitesimal transformation $\bar{x}^i \approx x^i + a\xi^i(x)$ of this group and using the rule for differentiating Jacobians,

$$\frac{\partial J}{\partial a} \Big|_{a=0} = D_i(\xi^i),$$

we obtain the infinitesimal transformation (5.57):

$$\bar{f} \approx f - a D_i(\xi^i) f. \quad (5.59)$$

For the δ -function this has the form

$$\bar{\delta} \approx \delta - a \left[D_i(\xi^i) \Big|_{x=0} \right] \delta. \quad (5.60)$$

The Leray form. Let us consider a hypersurface in \mathbb{R}^n defined by the equation

$$P(x) = 0 \quad (5.61)$$

with a continuously differentiable function $P(x)$, where $\text{grad} P \neq 0$ on the surface $P = 0$. The *Leray form* ([107], Ch. IV, §1; see also [55], Ch. III, §1, [31], p. 439) for this hypersurface is defined as an $(n-1)$ -form ω such that

$$dP \wedge \omega = dx^1 \wedge \dots \wedge dx^n.$$

At the points where $P_j \equiv \frac{\partial P}{\partial x^j} \neq 0$ it can be represented in the form

$$\omega = (-1)^{j-1} \frac{dx^1 \wedge \dots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \dots \wedge dx^n}{P_j}.$$

Using the Leray form on the surface (5.61) one can define the *Dirac measure* $\delta(P)$ by

$$\langle \delta(P), \varphi \rangle = \int_{P=0} \varphi \omega. \quad (5.62)$$

Let $\theta(p)$ be the Heaviside function, or the characteristic function of the domain $P \geq 0$, i.e.

$$\theta(P) = \begin{cases} 0 & \text{for } P < 0, \\ 1 & \text{for } P \geq 0. \end{cases}$$

The distribution

$$\langle \theta(P), \varphi \rangle = \int_{P \geq 0} \varphi(x) dx \quad (5.63)$$

satisfies the differential equation

$$\theta'(P) = \delta(P). \quad (5.64)$$

Differential equations with distributions. Here we do not deal with the problem of describing all solutions of arbitrary linear differential equations in the space of distributions (in this regard, see, e.g. [55], Ch. I. §2.6 or [31], pp. 450-452). We dwell only on those simplest equations that are needed in what follows. One of them is (5.64).

Let us take $n = 1$ and consider the first-order equation

$$x f' = 0. \quad (5.65)$$

Its only classical solution is $f = \text{const.}$ In distributions it has two linearly independent solutions: $f_1 = 1$ and $f_2 = \theta(x)$. Therefore the general solution of (5.65) in the space of distributions has the form

$$f = C_1 \theta(x) + C_2. \quad (5.66)$$

The Dirac measure (5.62) satisfies the following equations:

$$P \delta(P) = 0, \quad (5.67)$$

$$P \delta'(P) + \delta(P) = 0, \quad (5.68)$$

as well as the equations of order $m = 2, 3, \dots$, namely:

$$P \delta^{(m)}(P) + m \delta^{(m-1)}(P) = 0. \quad (5.69)$$

Indeed, equation (5.67) follows from the definition (5.62):

$$\langle P \delta(P), \varphi \rangle = \langle \delta(P), P \varphi \rangle = \int_{P=0} P \varphi \omega = 0.$$

Furthermore, if $\frac{\partial P}{\partial x^i} \neq 0$ (for some i it must be true in view of the condition $\text{grad} P \neq 0$), then differentiation of (5.67) with respect to x^i leads to the equation

$$\frac{\partial P}{\partial x^i} \delta(P) + P \delta'(P) \frac{\partial P}{\partial x^i} = 0,$$

which on division by $\frac{\partial P}{\partial x^i}$ yields (5.68). Further differentiation leads to the equations (5.69).

It follows from (5.69) that the first-order equation

$$P f'(P) + m f(P) = 0 \quad (5.70)$$

has the general solution

$$f = C_1 \delta^{(m-1)}(P) + C_2 P^{-m}. \quad (5.71)$$

§ 5.4.2 The Laplace equation

It is well known that Laplace's equation

$$\Delta u = 0$$

is conformally invariant (moreover, it is the only conformally invariant linear *elliptic* second-order equation [86]) and admits the Lie algebra L spanned by

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i}, & X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \\ Y_i &= (2x^i x^j - |x|^2 \delta^{ij}) \frac{\partial}{\partial x^j} + (2-n)x^i u \frac{\partial}{\partial u}, \\ Z_1 &= x^i \frac{\partial}{\partial x^i}, & Z_2 &= u \frac{\partial}{\partial u} \quad (i, j = 1, \dots, n). \end{aligned} \quad (5.72)$$

We shall regard the equation of a fundamental solution,

$$\Delta u = \delta(x), \quad (5.73)$$

as a boundary-value problem, in which at a fixed point (at the origin) we are given a δ -function singularity. Let us apply to this problem the invariance principle. To that end, we first single out in L all operators leaving invariant the singular point $x = 0$. It is easy to see that this property is shared by all operators (5.72) apart from the translation operators X_i ($i = 1, \dots, n$). For the rotation operators X_{ij} and for the stretchings Z_1 and Z_2 it is obvious, whereas the validity of the invariance condition $Y_i x^k|_{x=0} = 0$ of the point $x = 0$ under Y_i is clear from the equation $Y_i x^k = 2x^i x^k - |x|^2 \delta^{ik}$.

Let us return now to the equation (5.73). First of all, it admits the operators X_{ij} , since the Laplacian and the δ -function are invariant with respect to rotations (see (5.58)). As far as the operators Z_1 and Z_2 are concerned, there exists an admitted linear combination of these operators:

$$Z = x^i \frac{\partial}{\partial x^i} + ku \frac{\partial}{\partial u}, \quad k = \text{const.} \quad (5.74)$$

To find k , one has to satisfy the infinitesimal invariance criterion of equation (5.73). For that one has to prolong the operator (5.74) to second derivatives and add the transformation of the δ -function in accordance with (5.60). Since in this case $D_i(\xi^i) = n$, the resulting operator has the form

$$\tilde{Z} = x^i \frac{\partial}{\partial x^i} + ku \frac{\partial}{\partial u} + (k-1)u_i \frac{\partial}{\partial u_i} + (k-2)u_{ij} \frac{\partial}{\partial u_{ij}} - n\delta \frac{\partial}{\partial \delta}. \quad (5.75)$$

Therefore

$$\tilde{Z}(\Delta u - \delta) = (k-2)\Delta u + n\delta,$$

and the invariance condition of equation (5.73) is written

$$\tilde{Z}(\Delta u - \delta)|_{\Delta u=0} = (k-2+n)\delta = 0,$$

whence $k = 2 - n$. Thus equation (5.73) admits the following operators:

$$X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \quad Z = x^i \frac{\partial}{\partial x^i} + (2-n)u \frac{\partial}{\partial u}. \quad (5.76)$$

Likewise one can verify that the operators Y_i are also admitted. However, we do not need them and regard them as an *excess symmetry* of the fundamental solution. In accordance with the invariance principle, we shall look for a solution of equation (5.73) that is invariant with respect to the group with the basic generators (5.76). The rotation group has two invariants: $r = \sqrt{(x^1)^2 + \dots + (x^n)^2}$ and u . In these invariants the generator Z (see (5.76)) of the dilation group is written in the form

$$Z = r \frac{\partial}{\partial r} + (2-n)u \frac{\partial}{\partial u}$$

and supplies us with the invariant

$$J = ur^{n-2}$$

common to all operators (5.76). By Theorem 21.3 an invariant solution can be written in the form $J = \text{const.}$, i.e.

$$u = Cr^{2-n}, \quad C = \text{const.} \quad (5.77)$$

Substituting (5.77) into (5.73) we find

$$C = \frac{1}{(2-n)\Omega_n},$$

where Ω_n is the surface area of the unit sphere in the n -dimensional space. Hence, the fundamental solution has the form

$$u = \frac{1}{(2-n)\Omega_n} r^{2-n}. \quad (5.78)$$

Note that the fundamental solution (5.78) has been determined from the invariance condition up to a constant factor, and equation (5.73) played the role of a normalizing condition only.

§ 5.4.3 The heat equation

In this case equation (5.73) is replaced by the equation

$$u_t - \Delta u = \delta(t, x), \quad (5.79)$$

where Δ is the n -dimensional Laplacian in the variables x^i , and the operators (5.76) are replaced by the following generators of rotations, Galilean transformations and dilations:

$$\begin{aligned} X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \quad Y_i = 2t \frac{\partial}{\partial x^i} - x^i u \frac{\partial}{\partial u}, \\ Z &= 2t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} - nu \frac{\partial}{\partial u}. \end{aligned} \quad (5.80)$$

As in the case of Laplace's equation, the operators (5.80) are singled out from the general symmetry algebra of the heat equation by requiring the invariance of equation (5.79).

Now we use the invariance principle. The invariants for X_{ij} are t , r , u . In the space of these invariants the operators Y_i are written in the form

$$Y_i = x^i \left(2 \frac{t}{r} \frac{\partial}{\partial r} - u \frac{\partial}{\partial u} \right).$$

Therefore the rotations and the Galilean transformations have two common invariants, t and $p = e^{r^2/(4t)}$. Rewriting the last of the operators (5.80) in terms of these invariants we have

$$Z = 2t \frac{\partial}{\partial t} - np \frac{\partial}{\partial p}$$

and obtain the following invariant:

$$J = u \sqrt{t^n} e^{\frac{r^2}{4t}}$$

Therefore the invariant solution is of the form

$$u = \frac{C}{\sqrt{t^n}} e^{-\frac{r^2}{4t}}. \quad (5.81)$$

Again, equation (5.79) plays the role of a normalizing condition. Namely, substituting the function (5.81) into (5.79) we determine the value of the constant C and find the fundamental solution of the heat equation:

$$u = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{r^2}{4t}}, \quad t > 0. \quad (5.82)$$

§ 5.4.4 The wave equation

Reasoning as in Section § 5.4.2, one can verify that the equation

$$u_{tt} - \Delta u = \delta(t, x) \quad (5.83)$$

admits the following operators:

$$\begin{aligned} X_{ij} &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad Y_i = t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t}, \\ Z &= t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} + (1-n)u \frac{\partial}{\partial u}, \quad i, j = 1, \dots, n. \end{aligned} \quad (5.84)$$

The operators (5.84) can be obtained from the $(n+1)$ -dimensional version of (5.76) by making one of the variables imaginary. It is manifest from this analogy that the rotations X_{ij} and Lorentz transformations Y_i furnish two invariants, u and $\tau = t^2 - r^2$, and that in these invariants the last operator of (5.84) is written

$$Z = 2\tau \frac{\partial}{\partial \tau} + (1-n)u \frac{\partial}{\partial u}. \quad (5.85)$$

It is also clear that the invariant common to all the operators (5.84) is

$$J = u\tau^{(n-1)/2}.$$

Therefore, one may seek the invariant solution in the form

$$u = C\tau^{(1-n)/2}. \quad (5.86)$$

However, substitution of (5.86) into (5.83) shows that there is no such solution. The reason is very simple: equation (5.83) has no classical solutions.

Nonetheless, we shall find an invariant solution by moving into the space of distributions. To this end, let us look for an invariant solution of the form

$$u = f(\tau). \quad (5.87)$$

Equation (5.87) admits the operators X_{ij} and Y_i since it is written in terms of the invariants u and τ of these operators. Therefore, it remains to examine the invariance condition with respect to the operator (5.85). This condition is written

$$Z(f(\tau) - u)|_{u=f} = 0,$$

or

$$2\tau f'(\tau) + (n-1)f(\tau) = 0. \quad (5.88)$$

The classical solution of the ordinary differential equation (5.88) is given by (5.86). Let us find the general solution of (5.88) in the space of distributions.

It suffices to treat the case of odd n , since the fundamental solution for even n can be obtained by Hadamard's method of descent. Thus, let $n = 2m + 1$, where $m = 0, 1, \dots$. Then equation (5.88) is written

$$\tau f'(\tau) + mf(\tau) = 0 \quad (5.89)$$

and coincides with equation (5.70). Therefore its general solution for $m = 0$ is given by (5.66):

$$f(\tau) = C_1\theta(\tau) + C_2, \quad (5.90)$$

and for $m \neq 0$ is given by (5.71):

$$f(\tau) = C_1\delta^{(m-1)}(\tau) + C_2\tau^{-m}. \quad (5.91)$$

Substitution into (5.83) yields

$$2C_1 = \pi^{-m}, \quad C_2 = 0.$$

Returning to the original notation, we obtain a fundamental solution of the wave equation for any odd number n of spatial variables:

$$u = \frac{1}{2}\theta(t^2 - x^2) \quad \text{for } n = 1, \quad (5.92)$$

$$u = \frac{1}{2}\pi^{(1-n)/2}\delta^{(\frac{n-3}{2})}(t^2 - |x|^2) \quad \text{for } n \geq 3. \quad (5.93)$$

The group method of constructing fundamental solutions presented above was applied recently [18] to wave equations in Lorentz spaces with a non-trivial conformal group.

CHAPTER 6

Summary of results

Table 1. Frequently encountered groups on the plane

Group	Generator	Invariant	Canonical variables in which $X = \frac{\partial}{\partial t}$
Translations in x : $\bar{x} = x + a$, $\bar{y} = y$ in y : $\bar{x} = x$, $\bar{y} = y + a$ along the straight line $kx + ly = 0$: $\bar{x} = x + la$, $\bar{y} = y - ka$	$X = \frac{\partial}{\partial x}$ $X = \frac{\partial}{\partial y}$ $X = l \frac{\partial}{\partial x} - k \frac{\partial}{\partial y}$	$J = y$ $J = x$ $J = kx + ly$	$t = x$, $u = y$ $t = y$, $u = x$ $t = \frac{x}{l}$, $u = kx + ly$
Rotation $\bar{x} = x \cos a + y \sin a$ $\bar{y} = y \cos a - x \sin a$	$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$	$J = x^2 + y^2$	$t = \arctan \frac{x}{y}$, $u = \sqrt{x^2 + y^2}$
Lorentz transformation $\bar{x} = x \cosh a + y \sinh a$ $\bar{y} = y \cosh a + x \sinh a$	$X = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$	$J = y^2 - x^2$	$t = \frac{1}{2} \ln \frac{y+x}{y-x}$, $u = y^2 - x^2$

Table 1 (continued)

Group	Generator	Invariant	Canonical variables in which $X = \frac{\partial}{\partial t}$
Galilean transformation $\bar{x} = x + ay, \bar{y} = y$	$X = y \frac{\partial}{\partial x}$	$J = y$	$t = \frac{x}{y}, u = y$
Homogeneous dilation $\bar{x} = xe^a, \bar{y} = ye^a$	$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	$J = \frac{x}{y}$	$t = \ln x, u = \frac{x}{y}$
Non-homogeneous dilation $\bar{x} = xe^a, \bar{y} = ye^{ka}$	$X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}$	$J = \frac{x^k}{y}$	$t = \ln x, u = \frac{x^k}{y}$
Projective transformation $\bar{x} = \frac{x}{1-ax}, \bar{y} = \frac{y}{1-ay}$	$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$	$J = \frac{x}{y}$	$t = -\frac{1}{x}, u = \frac{x}{y}$

Table 2. First-order equations with a known admitted operator

(I)	$y' = F(y)$	$X = \frac{\partial}{\partial x}$
	$y' = F(x)$	$X = \frac{\partial}{\partial y}$
	$y' = F(kx + ly)$	$X = l \frac{\partial}{\partial x} - k \frac{\partial}{\partial y}$
(II)	$y' = \frac{y + xF(r)}{x - yF(r)}, \quad r = \sqrt{x^2 + y^2}$	$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$
(III)	$y' = F(y/x)$	$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$
(IV)	$y' = x^{k-1} F(y/x^k)$	$X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}$
(V)	$xy' = F(xe^{-y})$	$X = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$
(VI)	$y' = yF(ye^{-x})$	$X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$
(VII)	$y' = \frac{y}{x} + xF(y/x)$	$X = \frac{\partial}{\partial x} + \frac{y}{x} \frac{\partial}{\partial y}$
(VIII)	$xy' = y + F(y/x)$	$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$
(IX)	$y' = \frac{y}{x + F(y/x)}$	$X = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$
(X)	$y' = \frac{y}{x + F(y)}$	$X = y \frac{\partial}{\partial x}$
(XI)	$xy' = y + F(x)$	$X = x \frac{\partial}{\partial y}$
(XII)	$xy' = \frac{y}{\ln x + F(y)}$	$X = xy \frac{\partial}{\partial x}$
(XIII)	$xy' = y[\ln y + F(x)]$	$X = xy \frac{\partial}{\partial y}$
(XIV)	$y' = P(x)y + Q(x)$	$X = e^{\int P(x)dx} \frac{\partial}{\partial y}$
(XV)	$y' = P(x)y + Q(x)y^n$	$X = y^n e^{(1-n) \int P(x)dx} \frac{\partial}{\partial y}$
(XVI)	$y' = P(x)y$	$X = y \frac{\partial}{\partial y}$

Table 3. Second-order equations with a known admitted operator

(I)	$y'' = F(y, y')$	$X = \frac{\partial}{\partial x}$
	$y'' = F(x, y')$	$X = \frac{\partial}{\partial y}$
	$y'' = F(kx + ly, y')$	$X = l \frac{\partial}{\partial x} - k \frac{\partial}{\partial y}$
(II)	$y'' = (1 + y'^2)^{3/2} F\left(r, \frac{y - xy'}{x + yy'}\right)$	$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$
(III)	$y'' = y'^3 F\left(y, \frac{y - xy'}{y'}\right)$	$X = y \frac{\partial}{\partial x}$
(IV)	$y'' + \frac{q''(y)}{q(y)} xy'^3 = y'^3 F\left(y, \frac{1}{y'} - \frac{q'(y)}{q(y)} x\right)$	$X = q(y) \frac{\partial}{\partial x}$
(V)	$y'' = F(x, y - xy')$	$X = x \frac{\partial}{\partial y}$
(VI)	$p(x)y'' - p''(x)y = F(x, p(x)y' - p'(x)y)$	$X = p(x) \frac{\partial}{\partial y}$
(VII)	$p^2(x)y'' + p(x)p'(x)y' = F(y, p(x)y')$	$X = p(x) \frac{\partial}{\partial x}$
(VIII)	$xy'' = F\left(\frac{y}{x}, y'\right)$	$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$
(IX)	$y'' = x^{k-2} F(x^{-k}y, x^{1-k}y')$	$X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}$
(X)	$y'' = yF(ye^{-x}, y'/y)$	$X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$
(XI)	$y'' = yF(x, y'/y)$	$X = y \frac{\partial}{\partial x}$
(XII)	$y'' = \frac{q'(y)}{q(y)} y'^2 + q(y) F\left(x, \frac{y'}{q(y)}\right)$	$X = q(y) \frac{\partial}{\partial y}$
(XIII)	$yy'' = y'^2 + y^2 F\left(x, \frac{xy'}{y} - \ln y\right)$	$X = xy \frac{\partial}{\partial y}$
(XIV)	$xy'' + y' = x^2 y'^3 F\left(y, \frac{y}{xy'} - \ln x\right)$	$X = xy \frac{\partial}{\partial x}$
(XV)	$x^3 y'' = F\left(\frac{y}{x}, y - xy'\right)$	$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$
(XVI)	$x^3 y'' = y'^3 F\left(\frac{y}{x}, \frac{y - xy'}{y'}\right)$	$X = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$
(XVII)	$x^{k+1} [xy'' + (1 - k)y'] = F\left(\frac{y}{x^k}, xy' - ky\right)$	$X = x^k \left(x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}\right)$
(XVIII)	$xy [yy'' + (k - 1)y'^2] = y'^3 F\left(\frac{x}{y^k}, \frac{y}{y'} - kx\right)$	$X = y^k \left(kx \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$

Table 4. Canonical form of second-order equations
admitting a two-dimensional Lie algebra

Type	Structure of L_2	Basis of L_2 in canonical variables	Equation
I	$[X_1, X_2] = 0, X_1 \vee X_2 \neq 0$	$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}$	$y'' = f(y')$
II	$[X_1, X_2] = 0, X_1 \vee X_2 = 0$	$X_1 = \frac{\partial}{\partial y}, X_2 = x \frac{\partial}{\partial y}$	$y'' = f(x)$
III	$[X_1, X_2] = X_1, X_1 \vee X_2 \neq 0$	$X_1 = \frac{\partial}{\partial y}, X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	$y'' = \frac{1}{x} f(y')$
IV	$[X_1, X_2] = X_1, X_1 \vee X_2 = 0$	$X_1 = \frac{\partial}{\partial y}, X_2 = y \frac{\partial}{\partial y}$	$y'' = f(x)y'$

Table 5. Computation of Lie algebras of operators
admitted by second-order equations

Differential equation (1.31)	$y'' = f(x, y, y')$
Admitted operator	$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$
Second prolongation of the operator	$X_{(2)} = X + \zeta_1 \frac{\partial}{\partial y'} + \zeta_2 \frac{\partial}{\partial y''},$ $\zeta_1 = D(\eta) - y' D(\xi) \equiv \eta_x + (\eta_y - \xi_x) y' - y'^2 \xi_y,$ $\zeta_2 = D(\zeta_1) - y'' D(\xi) \equiv \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - y'^3 \xi_{yy} + (\eta_y - 2\xi_x - 3y' \xi_y) y''$
Determining equation (1.32)	$\eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - y'^3 \xi_{yy}$ $+ (\eta_y - 2\xi_x - 3y' \xi_y) f - \xi f_x - \eta f_y$ $- [\eta_x + (\eta_y - \xi_x) y' - y'^2 \xi_y] f_{y'} = 0$

Table 6. Algorithm for integration of second-order equations using a two-dimensional Lie algebra L_2 by the method of canonical variables

Step	Actions	Result
1	Find the algebra L_r of all admitted operators $X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$ by solving determining equation (1.32) of Table 5	Basis of $L_r : X_1, \dots, X_r$
2	If $r = 2$ go to next step, if $r > 2$ find in L_r a subalgebra L_2 using Theorem 21.9. For $r = 1$ the order of the equation can be reduced (§ 1.4), while for $r = 0$ the group-theoretical approach is inapplicable	Basis of $L_2 : X_1, X_2$
3	Determine the type of L_2 by Theorem 21.11: compute the commutator $[X_1, X_2]$ and the pseudo-scalar product $X_1 \vee X_2$ (see (2.8)). If $[X_1, X_2] \neq 0$ and is not equal to X_1 , then by the linear change of variables (2.10) pass to a new basis X'_1, X'_2 with $[X'_1, X'_2] = X'_1$	Reduction of the structure of L_2 to one of the four types of Table 4
4	By Theorem 21.12, reduce the basis of L_2 to one of Table 4 by a suitable change of variables x, y . By this change of variables reduce your equation to canonical form of corresponding type from Table 4 and integrate it (see § 2.2.3)	Determination of the integrating change of variables
5	Rewrite the solution in original variables	Solution of the equation

Table 7. Real non-isomorphic structures of three-dimensional Lie algebras (classification by Bianchi [22])

Type	$[X_1, X_2]$	$[X_2, X_3]$	$[X_3, X_1]$
I	0	0	0
II	0	0	$-X_2$
III	0	0	$-X_1$
IV	0	X_2	$-(X_1 + X_2)$
V	0	X_2	$-X_1$
VI	0	$kX_2, k \neq 0 \text{ and } 1$	$-X_1$
VII	0	$X_1 + qX_2$	$X_2 - qX_1$
VIII	X_1	X_3	$-2X_2$
IX	X_3	X_1	X_2
<p><i>Remark.</i> In complex domain there are only seven non-isomorphic structures: type VII becomes VI with $k = \frac{q+i}{q-i}$ in the complex basis $\bar{X}_1 = X_2 + iX_1$, $\bar{X}_2 = X_2 - iX_1$, $\bar{X}_3 = \frac{1}{q-i} X_3$, and type IX coincides with VII in the basis $\bar{X}_1 = X_1 - iX_3$, $\bar{X}_2 = -iX_2$, $\bar{X}_3 = X_1 + iX_3$.</p>			

Table 8. Real non-similar Lie algebras L_3 and invariant equations

N	Type in Table 7	Basis of L_3	Equation
1*	IX	$X_1 = (1 + x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y},$ $X_2 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},$ $X_3 = xy \frac{\partial}{\partial x} + (1 + y^2) \frac{\partial}{\partial y}$	$y'' = C \left[\frac{1 + y^2 + (1 + x^2)y'^2 - 2xyy'}{1 + x^2 + y^2} \right]^{3/2}$
2	VIII	$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$ $X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$ $X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$	$y'' = 2 \frac{y' + Cy'^{3/2} + y'^2}{y - x}$
3		$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial y},$ $X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$	$y'' = Cy^{-3}$
4**		$X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y},$ $X_3 = y^2 \frac{\partial}{\partial y}$	$y''' = \frac{3}{2} \frac{y''^2}{y'} + f(x)y'$
5*	VII	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$ $X_3 = (qx + y) \frac{\partial}{\partial x} + (qy - x) \frac{\partial}{\partial y}$	$y'' = C(1 + y'^2)^{3/2} e^{q \arctan y'}$
6	VI if $k \neq 0$	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$ $X_3 = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}, \quad k \neq 1$	$y'' = Cy' \frac{k-2}{k-1}$
7	III if $k = 0$	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y},$ $X_3 = (1 - k)x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad k \neq 1$	$y'' = Cx \frac{1-2k}{k-1}$

Table 8 (continued)

N	Type in Table 7	Basis of L_3	Equation
8	V	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$ $X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	$y'' = 0$
9		$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial y}$	
10	IV	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$ $X_3 = x \frac{\partial}{\partial x} + (x + y) \frac{\partial}{\partial y}$	$y'' = Ce^{-y'}$
11		$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y},$ $X_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	$y'' = Ce^x$
12	II	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial y}$	$y'' = C$
13**	I	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y},$ $X_3 = p(x) \frac{\partial}{\partial y}, \quad p''(x) \neq 0$	$y''' = \frac{p'''(x)}{p''(x)} y'' + f(x)$
<p><i>Remarks.</i> * In complex domain, case 1 reduces to case 2 by changing to variables \bar{x}, \bar{y} defined by the equations $x = \frac{\bar{x}\bar{y} - 1}{\bar{x} + \bar{y}}$, $y = -i \frac{\bar{x}\bar{y} + 1}{\bar{x} + \bar{y}}$, while 5 reduces to 6 with $k = \frac{q+i}{q-i}$ by the change of variables $\bar{x} = \frac{1}{2}(y - ix)$, $\bar{y} = \frac{1}{2}(y + ix)$.</p> <p>** In cases 4 and 13 there are no invariant second-order equations, therefore the third-order equations are given.</p>			

Table 9. Lie's group classification of second-order equations

Group	Symmetries	Equation
G_1	$X_1 = \frac{\partial}{\partial x}$	$y'' = f(y, y')$
G_2	$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}$	$y'' = f(y')$
	$X_1 = \frac{\partial}{\partial y}, X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	$y'' = \frac{1}{x} f(y')$
G_3	$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$ $X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$	$y'' = 2 \frac{y' + C y'^{3/2} + y'^2}{y - x}$
	$X_1 = \frac{\partial}{\partial x}, X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$ $X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$	$y'' = C y^{-3}$
	$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}, X_3 = x \frac{\partial}{\partial x} + (x + y) \frac{\partial}{\partial y}$	$y'' = C e^{-y'}$
	$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}, X_3 = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}$	$y'' = C y'^{\frac{k-2}{k-1}},$ $k \neq 0, \frac{1}{2}, 1, 2$
G_8	$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}, X_3 = x \frac{\partial}{\partial y}, X_4 = x \frac{\partial}{\partial x},$ $X_5 = y \frac{\partial}{\partial x}, X_6 = y \frac{\partial}{\partial y}, X_7 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y},$ $X_8 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$	$y'' = 0$

Table 10. Construction of the algebra admitted by a linear hyperbolic equation in two variables

Equation	$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0$
Laplace invariants	$h = a_x + ab - c, \quad k = b_y + ab - c$
The general form of an admitted operator	$X = \xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} + \sigma(x, y)u \frac{\partial}{\partial u}$
Differential form dual to the operator X	$\Xi = \eta dx - \xi dy$
Determining equations	$d(h\Xi) = 0, \quad d(k\Xi) = 0, \quad d(\sigma + a\eta + b\xi) = (h - k)\Xi$
Coordinate form of the determining equations	$(h\xi)_x + (h\eta)_y = 0, \quad (k\xi)_x + (k\eta)_y = 0,$ $(\sigma + a\eta + b\xi)_x = (h - k)\eta,$ $(\sigma + a\eta + b\xi)_y = (k - h)\xi$

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Bibliography

- [1] I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov. Group classification of the equations of nonlinear filtration. *Dokl. Akad. Nauk SSSR*, 293, No. 5:1033–1036, 1987. English transl., *Soviet Math. Dokl.*, 35(2), (1987), pp. 384–386.
- [2] W. F. Ames. *Nonlinear partial differential equations in engineering*, volume I. Academic Press, New York, 1965.
- [3] W. F. Ames. *Nonlinear partial differential equations in engineering*, volume II. Academic Press, New York, 1972.
- [4] W. F. Ames. Nonlinear superposition for operator equations. In *Nonlinear equations in abstract spaces*, Ed. V. Lakshmikantham, pages 43–66. Academic Press, New York, 1978.
- [5] R. L. Anderson. A nonlinear superposition principle admitted by coupled Riccati equations of the projective type. *Lett. Math. Phys.*, 4:1–7, 1989.
- [6] R. L. Anderson and N. H. Ibragimov. Bianchi-Lie, Bäcklund and Lie-Bäcklund transformations. In *Proceedings: Group theoretical methods in mechanics*, Eds. N.H. Ibragimov and L.V. Ovsyannikov, pages 34–46. Institute of Hydrodynamics, USSR Acad. Sci., Siberian Branch, Novosibirsk, 1978.
- [7] R. L. Anderson and N. H. Ibragimov. *Lie-Bäcklund transformations in applications*. SIAM, Philadelphia, 1979.
- [8] R. L. Anderson, S. Kumei, and C. E. Wulfman. Generalizations of the concept of invariance of differential equations. *Phys. Rev. Letters*, 28:988, 1972.
- [9] R. L. Anderson and P. Winternitz. A nonlinear superposition principle for Riccati equations of the conformal type. In *Proceedings: Group*

- theoretical methods in physics*, pages 165–169. Mexico, Cocoyoc, 1980. Published in "Lecture Notes in Physics", No. 135, Springer-Verlag, New-York, 1980.
- [10] V. Anisimov. *A course of ordinary differential equations*. Warsaw, 1906.
 - [11] L. Asgeirsson. Some hints on Huygens' principle and Hadamard's conjecture. *Comm. Pure Appl. Math.*, 9, No. 3:307–327, 1956.
 - [12] A. V. Bäcklund. Einiges über Curven- und Flächen-Transformationen. *Lunds Universitets Års-skrift. II. Afdelningen för Matematik och Naturvetenskap*, X:1–12, 1873.
 - [13] A. V. Bäcklund. Ueber Flächentransformationen. *Mathematische Annalen*, IX:297–320, 1876.
 - [14] A. V. Bäcklund. Zur Theorie der partiellen Differentialgleichungen erster Ordnung. *Mathematische Annalen*, XVII:285–328, 1880.
 - [15] A. V. Bäcklund. Om ytor med konstant negativ krökning. *Lunds Universitets Års-skrift*, XIX:1–48, 1883. (Swedish). English transl., "Surfaces of constant negative curvature", *Archives of ALGA*, Vol. 2, 2005, pp. 7–44.
 - [16] H. Bateman. The transformation of the electrodynamical equations. *Proc. London Math. Soc., Ser. 2*, 8:223–264, 1910.
 - [17] E. Beltrami. Sul principio di Huygens. *Rend. Reale Istituto Lombardo*, Ser. 2, 22:428–438, 1889.
 - [18] Yu.Yu. Berest. Construction of fundamental solutions for Huygens' equations as invariant solutions. *Dokl. Akad. Nauk SSSR*, 317, No.4:786–789, 1991. English transl., Soviet Math. Dokl. 43 (1991), pp.496–499.
 - [19] L. M. Berkovich and N. Kh. Rozov. Some remarks on differential equations of the form $y'' + a_0(x)y = \varphi(x)y^\alpha$. *Differential Equations*, 8, No. 11:1609–1612, 1972.
 - [20] E. Bessel-Hagen. Über die Erhaltungssätze der Elektrodynamik. *Mathematische Annalen*, 84:258–276, 1921.
 - [21] L. Bianchi. Ricerche sulle superficie a curvatura costante e sulle eliocoidi. *Ann. Scuola Norm. Sup., Pisa*, II:285, 1879.

- [22] L. Bianchi. *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*. Enrico Spoerri, Pisa, 1918.
- [23] L. Bianchi. *Lezioni di Geometria Differenziale*, volume I. Enrico Spoerri, Pisa, 1922.
- [24] G. Birkhoff. *Hydrodynamics*. Princeton Univ. Press, Princeton, N.J., 2nd edition, 1960.
- [25] G. W. Bluman and S. Kumei. On the remarkable nonlinear diffusion equation $\frac{\partial}{\partial x} [a(u+b)^{-2} \frac{\partial u}{\partial x}] - \frac{\partial u}{\partial t} = 0$. *J. Math. Phys.*, 21, No. 5:1019–1023, 1980.
- [26] G. W. Bluman and S. Kumei. *Symmetries and Differential Equations*. Springer-Verlag, New-York, 1989.
- [27] N. N. Bogolyubov and D. V. Shirkov. *Introduction to the theory of quantized fields*. Gostekhizdat, 1957. (Russian).
- [28] M. Boiti, C. Laddomada, and F. Pempinelli. An equivalent real form of the nonlinear Schrödinger equation and the permutability for Bäcklund transformations. *Nuo Cimento*, 62B:315–326, 1981.
- [29] E. Candotti, C. Palmieri, and B. Vitale. On the inversion of Noether's theorem in the Lagrangian formalism. *Nuovo Cimento*, 70A:233–246, 1970.
- [30] N. G. Chebotarev. *Theory of Lie groups*. GITTL, M-L, 1940. (Russian).
- [31] Y. Choquet-Bruhat and C. DeWitt-Morette. *Analysis, manifolds and physics, Part I: Basics*. North-Holland, Amsterdam, 2nd edition, 1982. (Part II: 92 Applications.- North Holland, Amsterdam, 1989).
- [32] Christiaan Huygens. *Traité de la lumière. Où sont expliquées les causes de ce qui luy arrive dans la reflexion et dans la refraction. Et particulièrement dans l'étrange refraction du cristal d'Island*. Pierre van der Aa, Leiden, 1690. Latest reprint, Dawsons of Pall Mall, London, 1966; also reprinted in *Oeuvres complètes*, vol. 19, Martinus Nijhoff, The Hague, 1937, pp. 451–548; English transl., *Treatise on light*, Macmillan, London, 1912. Reprinted by Dover, New York, 1962.
- [33] J. Clairin. Sur les transformations de Bäcklund. *Ann. Sci. École Norm. Sup.*, Ser. 3, 19, Supplément:3–63, 1902.

- [34] E. T. Copson. On the Riemann-Green function. *Arch. Rational Mech. Anal.*, 1:324–348, 1958.
- [35] R. Courant. *Partial differential equations*. Interscience, New York, 1962. Volume II of *Methods of mathematical physics* by R. Courant and D. Hilbert.
- [36] E. Cunningham. The principle of relativity in electrodynamics and an extension thereof. *Proc. London Math. Soc.*, Ser. 2, 8:77, 1910.
- [37] G. Darboux. *Leçons sur la Théorie Générale des Surfaces*, volume III. Gauthier-Villars, Paris, 1894.
- [38] L. E. Dickson. Differential equations from the group standpoint. *Ann. of Math.*, 25:287–378, 1924.
- [39] P. A. M. Dirac. Wave equations in conformal space. *Ann. of Math.*, 37(2):429–442, 1936.
- [40] A. Douglis. A criterion for the validity of Huygens' principle. *Comm. Pure Appl. Math.*, 9, No. 3:391–402, 1956.
- [41] B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko. *Modern Geometry: Methods and applications. Parts I, II*. Nauka, Moscow, 1979. (2nd ed. 1986). English transl., Springer-Verlag. Second English ed. 1992.
- [42] A. S. Eddington. *The mathematical theory of relativity*. Cambridge Univ. Press, Cambridge, 2nd edition, 1924.
- [43] D. G. B. Edelen. Invariance theory for nonlocal variational principles. *International Journal of Engineering Science*, I-IV, No. 9, 1971.
- [44] L. P. Eisenhart. Equivalent continuous groups. *Ann. of Math.*, Ser. 2, 33:665–670, 1932.
- [45] L. P. Eisenhart. *Continuous groups of transformations*. Princeton Univ. Press, Princeton, N.J., 1933.
- [46] L. P. Eisenhart. *Riemannian Geometry*. Princeton Univ. Press, Princeton, N.J., 2nd edition, 1949.
- [47] F. Engel and K. Faber. *Die Liesche Theorie der partiellen Differentialgleichungen erster Ordnung*. Leipzig, 1932.

- [48] V. P. Ermakov. Second order differential equations. Conditions of complete integrability. *Universitetskie Izvestiya*, Kiev, 20, No. 9:1–25, 1880. From: “Lectures on integration of differential equations”. (Russian).
- [49] V. A. Fock. Remark on the paper of F. I. Frankle “On the correctness of the Cauchy problem and properties of harmonic coordinates in general relativity”. *Uspekhi Matematicheskikh Nauk*, 11, No. 3(69):197–200, 1956.
- [50] V. A. Fock. *The theory of space, time and gravitation*. Fizmatgiz, Moscow, 1961. English transl., Pergamon Press, New York, 1959.
- [51] A. S. Fokas and R.L. Anderson. Group theoretical nature of Bäcklund transformations. *Let. Math. Phys.*, 3:117, 1979.
- [52] F. I. Frankle. On the correctness of the Cauchy problem and properties of harmonic coordinates in general relativity. *Uspekhi Matematicheskikh Nauk*, 11, No. 3(69):189–196, 1956.
- [53] V. I. Fushchich. Group properties of the differential equations of quantum mechanics. In *Problems of the Asymptotic Theory of Nonlinear Oscillations*, pages 238–246. Naukova Dumka, Kiev, 1977. (Russian).
- [54] I. M. Gel’fand and L. A. Dikii. Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equations. *Uspekhi Mat. Nauk*, 30, No. 5:67–100, 1975. English transl., Russian Math. Surveys 30, No. 5 (1975), pp. 77–113.
- [55] I. M. Gel’fand and G. E. Shilov. *Generalized functions*, volume 1. Fizmatgiz, Moscow, 1959. English translation by E. Saletan, Academic Press, N.Y., 1964.
- [56] E. Goursat. *Differential equations*. Ginn and Co, Boston, 1917. English translation by E.R. Hedrick and O. Dunkel of Goursat’s classical *Cours d’analyse mathématique*, vol. 2, Part II.
- [57] E. Goursat. *Le problème de Bäcklund. Memorial des sciences mathématiques*. Fasc. VI, Gauthier-Villars, Paris, 1925.
- [58] A. Guldberg. Sur les équations différentielles ordinaire qui possèdent un système fondamental d’intégrales. *C.R. Acad. Sci. Paris*, 116:964, 1893.

- [59] P. Günther. Zur Gültigkeit des Huygensschen Prinzips bei partiellen Differentialgleichungen vom normalen hyperbolischen Typus. *Berichte über Verh. Sächs. Akad. d. Wiss. zu Leipzig, Math.-Naturwiss. Kl.*, 100, Heft 2, 1952.
- [60] P. Günther. Ein Beispiel einer nichttrivialen Huygensschen Differentialgleichung mit vier unabhängigen Variablen. *Arch. Rational Mech. Anal.*, 18, No. 2:103–106, 1965.
- [61] J. Hadamard. *Lectures on Cauchy's problem in linear partial differential equations*. Yale University Press, New Haven, 1923. Revised French edition: *Le problème de Cauchy*, Paris, 1932.
- [62] J. Hadamard. Principe de Huygens et la prolongement analytique. *Bull. Soc. Math. de France*, 52:241–278, 1924.
- [63] J. Hadamard. On a simple case of wave diffusion. *Mat. Sb.*, 41, No. 3:404–407, 1934. (Russian).
- [64] J. Hadamard. The problem of diffusion of waves. *Ann. of Math., Ser. 2*, 43:510–522, 1942.
- [65] E. L. Hill. Hamilton's principle and conservation theorems of mathematical physics. *Rev. Mod. Phys.*, 23, No.3:253–260, 1951.
- [66] Z. V. Huhunashvili. The symmetry of the differential equations of field theory. *Izvestia Vuzov: Fizika*, No. 3:95–103, 1971. English transl., *Soviet Phys. J.*, 14 (1971).
- [67] N. H. Ibragimov. Classification of the invariant solutions to the equations for the two-dimensional transient-state flow of a gas. *Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki*, 7, No. 4:19–22, 1966. English transl., *Journal of Applied Mechanics and Technical Physics*, 7(4) (1966), pp. 11–13.
- [68] N. H. Ibragimov. *Group properties of some differential equations*. Nauka, Novosibirsk, 1967. (Russian).
- [69] N. H. Ibragimov. Generalized motions in Riemannian spaces. *Dokl. Akad. Nauk SSSR*, 178(1):27–30, 1968. English transl., *Soviet Math. Dokl.*, vol. 9 (1968), p. 21.
- [70] N. H. Ibragimov. Group properties of wave equations with zero mass. *Dokl. Akad. Nauk SSSR*, 178(3):566–568, 1968.

- [71] N. H. Ibragimov. On the group classification of differential equations of second order. *Dokl. Akad. Nauk SSSR*, 183, No. 2:274–277, 1968. English transl., Soviet Math. Dokl., 9, No. 6, (1968), pp. 1365–1369.
- [72] N. H. Ibragimov. Transformations preserving harmonic coordinates. *Dokl. Akad. Nauk SSSR*, 181, No. 5:1050–1053, 1968. English transl., Soviet Math. Dokl., 9, No. 4, (1968), pp. 976–979.
- [73] N. H. Ibragimov. Groups of generalized motions. *Dokl. Akad. Nauk SSSR*, 187, No. 1:25–28, 1969. English transl., Soviet Math. Dokl., 10, No. 4, (1969), pp. 780–784.
- [74] N. H. Ibragimov. Invariance of Dirac's equations. *Dokl. Akad. Nauk SSSR*, 185, No. 6:1226–1228, 1969. English transl., Soviet Math. Dokl., 10, No. 2, (1969), pp. 488–491.
- [75] N. H. Ibragimov. Invariant variational problems and conservation laws. *Teoreticheskaya i Matematicheskaya Fizika*, 1, No. 3:350–359, 1969. English transl., Theor. Math. Phys., 1, No. 3, (1969), 267–276.
- [76] N. H. Ibragimov. The wave equation in a Riemannian space. *Continuum Dynamics*, 1:36–47, 1969. Publisher: Institute of Hydrodynamics, USSR Acad. Sci., Siberian Branch, Novosibirsk. (Russian).
- [77] N. H. Ibragimov. Conformal invariance and Huygens' principle. *Dokl. Akad. Nauk SSSR*, 194, No. 1:24–27, 1970. English transl., Soviet Math. Dokl., 11, No. 5, (1970), pp.1153–1157.
- [78] N. H. Ibragimov. The Huygens principle. In *Some problems of mathematics and mechanics*, pages 159–170. Nauka, Leningrad, 1970. Dedicated to the 70th anniversary of M.A. Lavrentyev. English transl., Amer. Math. Soc. Transl., (2) 104 (1976), pp. 141–151.
- [79] N. H. Ibragimov. *Lie groups in some problems of mathematical physics*. Novosibirsk Univ. Press, Novosibirsk, 1972. (Russian).
- [80] N. H. Ibragimov. Conservation laws in hydrodynamics. *Dokl. Akad. Nauk SSSR*, 210, No. 6:1307–1309, 1973. English transl., Soviet Physics Dokl., 18 (1973/74).
- [81] N. H. Ibragimov. Conservation laws in continuum mechanics. *Proc. Sympos. Symmetry, Similarity and Group Theoretic Methods in Mechanics*, Ed. P.G. Glockner and M.C. Singh, University of Calgary, Calgary:63–82, 1974.

- [82] N. H. Ibragimov. Lie–Bäcklund groups and conservation laws. *Dokl. Akad. Nauk SSSR*, 230, No. 1:26–29, 1976. English transl., Soviet Math. Dokl., 17, No. 5, (1976), pp. 1242–1246.
- [83] N. H. Ibragimov. Group theoretical nature of conservation theorems. *Lett. Math. Phys.*, 1:423–428, 1977.
- [84] N. H. Ibragimov. On the theory of Lie–Bäcklund transformation groups. *Mat. Sb.*, 109, No. 2:229–253, 1979. English transl., Math. USSR Sbornik, 37, No 2 (1980), 205–226.
- [85] N. H. Ibragimov. Sur l'équivalence des équations d'évolution, qui admettent une algèbre de Lie–Bäcklund infinie. *C.R. Acad. Sci. Paris*, Sér. I, 293:657–660, 1981.
- [86] N. H. Ibragimov. *Transformation groups in mathematical physics*. Nauka, Moscow, 1983. English transl., *Transformation groups applied to mathematical physics*, Riedel, Dordrecht, 1985.
- [87] N. H. Ibragimov. *Primer of group analysis*. Znanie, No. 8, Moscow, 1989. (Russian). Revised edition in English: *Introduction to modern group analysis*, Tau, Ufa, 2000. Available also in Swedish: *Modern grouppanalis: En inledning till Lies lösningsmetoder av ickelinjära differentialekvationer*, Studentlitteratur, Lund, 2002.
- [88] N. H. Ibragimov. Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie). *Uspekhi Mat. Nauk*, 47, No. 4:83–144, 1992. English transl., Russian Math. Surveys, 47:2 (1992), 89–156.
- [89] N. H. Ibragimov. *Essay in group analysis of ordinary differential equations*. Znanie, Moscow, 7/1991. (Russian).
- [90] N. H. Ibragimov and R. L. Anderson. Groups of Lie–Bäcklund contact transformations. *Dokl. Akad. Nauk SSSR*, 227, No. 3:539–542, 1976. English transl., Soviet Math. Dokl., 17, No. 2, (1976), pp. 437–441.
- [91] N. H. Ibragimov and R. L. Anderson. Lie–Bäcklund tangent transformations. *J. Math. Anal. and Appl.*, 59, No. 1:145–162, 1977.
- [92] N. H. Ibragimov and E. V. Mamontov. Sur le problème de J. Hadamard relatif à la diffusion des ondes. *C.R. Acad. Sci. Paris*, Sér. A, 270:A456–A458, 1970.

- [93] N. H. Ibragimov and E. V. Mamontov. On the Cauchy problem for the equation $u_{tt} - u_{xx} - \sum_{i,j=1}^{n-1} a_{ij}(x-t)u_{y_i y_j} = 0$. *Mat. Sb.*, 102, No. 3:391–409, 1977. English transl., *Math. USSR Sbornik*, 31, No. 3, (1977), pp. 347–363.
- [94] N. H. Ibragimov and S. V. Meleshko. Linearization of third-order ordinary differential equations by point and contact transformations. *Journal of Mathematical Analysis and Applications*, 308, No. 1:266–289, 2005.
- [95] N. H. Ibragimov and A. O. Oganessian. The hierarchy of Huygens' equations in spaces with a non-trivial conformal group. *Uspekhi Mat. Nauk*, 46, No. 3:111–146, 1991. English transl., *Russian Math. Surveys*, 46:3 (1991), pp. 137–176.
- [96] N. H. Ibragimov and A. B. Shabat. Infinite Lie-Bäcklund algebras. *Funkts. Anal. Prilozhen.*, 14, No. 4:79–80, 1980. English transl., *Functional Anal. Appl.*, 14, No. 4 (1980), pp. 313–315.
- [97] E. L. Ince. *Ordinary differential equations*. Dover, New York, 1944.
- [98] D. D. Ivanenko, editor. *Nonlinear quantum field theory*. IL, Moscow, 1959. (Collection of papers translated into Russian).
- [99] C. G. J. Jacobi. *Vorlesungen über Dynamik*. Reimer, Berlin, 2nd edition, 1884.
- [100] F. Klein. *Review of Theorie der Transformationgruppen*. Bd. III, at the award of first N.I. Lobachevsky Prize, 22 October 1897. Typography of the Kazan Imperial University, Kazan, 1898.
- [101] F. Klein. Über die Differentialgesetze für die Erhaltung von Impuls und Energie in der Einsteinschen Gravitationstheorie. *Königliche Gesellschaft der Wissenschaften zu Göttingen, Nachrichten. Mathematisch-Physikalische Klasse*, Heft 3:171–189, 1918.
- [102] F. Klein. Über die Integralform der Erhaltungssätze und die Theorie der räumlich-geschlossenen Welt. *Königliche Gesellschaft der Wissenschaften zu Göttingen, Nachrichten. Mathematisch-Physikalische Klasse*, Heft 3:394–423, 1918.

- [103] J. Krause and L. Michel. *Classification of the symmetries of ordinary differential equations*. Proc. Seminar: Group theoretical methods in physics, Moscow, June 1990. Springer-Verlag, New York, 1991.
- [104] B. A. Kupershmidt. Lagrangian formalism in the calculus of variations. *Funkts. Anal. Prilozhen.*, 10, No. 2:77–78, 1976. English transl., Functional Anal. Appl. 10 (1976).
- [105] Masatake Kuranishi. On the local theory of continuous infinite pseudogroups. *I, II, Nagoya Math. J.*, 15:225–260, 1959. 19, (1961), 55–91.
- [106] E. V. Lenskii. Group properties of equations of motion of a non-linear viscoplastic medium. *Vestnik MGU: Matematika i Mekhanika*, No. 5:116–125, 1966. (Russian).
- [107] J. Leray. *Hyperbolic differential equations*. Lecture Notes, Institute for Advanced Study, Princeton, 1953. Available in a book form in Russian translation by N.H. Ibragimov, Nauka, Moscow, 1984.
- [108] S. Lie. Begründung einer Invariantentheorie der Berührungstransformationen. *Mathematische Annalen*, 8, Heft 2-3:215–303, 1874. In Ges. Abhandl., Bd. 4, Teubner, Leipzig – Aschehoug, Oslo, 1929. Reprinted by Johnson Reprint Corp., New York, 1973.
- [109] S. Lie. Zur Theorie des Integrabilitätsfactors. *Forhandlinger i Videnskabs-Selskabet i Christiania* (Abbr. *Christ. Forh.*), pages 242–254, 1874. Reprinted in Ges. Abhandl., Bd. 3, Teubner, Leipzig–Aschehoug, Oslo, 1922, paper XIII.
- [110] S. Lie. Zur Theorie der Flächen konstanter Krümmung, III. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 5, Heft 3:282–306, 1880. Reprinted in Ges. Abhandl., Bd. 3, pp. 398–420, Teubner, Leipzig–Aschehoug, Oslo, 1922; reprint, Johnson Reprint Corp., New York, 1973.
- [111] S. Lie. Zur Theorie der Flächen konstanter Krümmung, IV. bestimmung aller Flächen konstanter Krümmung durch successive quadraturen. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 5, Heft 3:328–358, 1880. Reprinted in Ges. Abhandl., Bd. 3, pp. 421–446, Teubner, Leipzig–Aschehoug, Oslo, 1922; reprint, Johnson Reprint Corp., New York, 1973.
- [112] S. Lie. Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 6, Heft 3:328–368,

1881. English transl., *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2: Applications in Engineering and Physical Sciences*, ed. N.H. Ibragimov, CRC Press, Boca Roton, 1995. Reprinted also in the book *Lie group analysis: Classical heritage*, ed. N.H. Ibragimov, ALGA Publications, Karlskrona, 2004.
- [113] S. Lie. Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y , die eine Gruppe von Transformationen gestatten. I. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 8, Heft 2-3:187–248, 1883. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper X, pp. 240–281.
- [114] S. Lie. Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y , die eine Gruppe von Transformationen gestatten. III. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 8, Heft 4:371–458, 1883. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XIV, pp. 362–427.
- [115] S. Lie. Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y , die eine Gruppe von Transformationen gestatten. II. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 8, Heft 3:249–288, 1883. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XI, pp. 249–310.
- [116] S. Lie. Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y , die eine Gruppe von Transformationen gestatten. IV. *Archiv for Matematik og Naturvidenskab* (Abbr. *Arch. for Math.*), 9, Heft 4:431–448, 1884. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XVI, pp. 432–446.
- [117] S. Lie. *Theorie der Transformationsgruppen*, Vol. I. (Bearbeitet unter Mitwirkung von F. Engel), B. G. Teubner, Leipzig, 1888.
- [118] S. Lie. Die infinitesimalen Berührungstransformationen der Mechanik. *Leipzig. Ber.*, 1889. Reprinted in Ges. Abhandl., Bd. 6, 1927, paper VI.
- [119] S. Lie. *Theorie der Transformationsgruppen*, Vol. II. (Bearbeitet unter Mitwirkung von F. Engel), B. G. Teubner, Leipzig, 1890.
- [120] S. Lie. *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*. (Bearbeitet und herausgegeben von Dr. G. Scheffers), B. G. Teubner, Leipzig, 1891.

- [121] S. Lie. *Theorie der Transformationsgruppen*, Vol. III. (Bearbeitet unter Mitwirkung von F. Engel), B. G. Teubner, Leipzig, 1893.
- [122] S. Lie. *Vorlesungen über continuerliche Gruppen mit geometrischen und anderen Anwendungen*. (Bearbeitet und herausgegeben von Dr. G. Scheffers), B. G. Teubner, Leipzig, 1893.
- [123] S. Lie. Zur allgemeine Theorie der partiellen Differentialgleichungen beliebiger Ordnung. *Leipzig. Ber.*, 1:53–128, 1895. Reprinted in Ges. Abhandl., Bd. 4, pp. 320–384. English transl. in: "Lie group analysis: classical heritage", ed. N.H. Ibragimov, ALGA publications, Karlskrona, Sweden, 2004, pp. 1–63.
- [124] S. Lie. Die infinitesimalen Berührungstransformationen der Optik. *Leipzig. Ber.*, Heft 1:131–133, 1896. Reprinted in Ges. Abhandl., Bd. 6, paper XXIV.
- [125] S. Lie. *Geometrie der Berührungstransformationen*. (Dargestellt von Sophus Lie und Georg Scheffers), B. G. Teubner, Leipzig, 1896.
- [126] S. Lie. *Gesammelte Abhandlungen*. B. G. Teubner (Leipzig)– H. Aschehoug and Co. (Oslo), Bd.1, 1934. Bd. 2 (Teil 1), 1935; Bd.2 (Teil 2), 1937; Bd.3, 1922; Bd.4, 1929; Bd.5, 1924; Bd. 6, 1927.
- [127] F. M. Mahomed and P.G.L. Leach. Lie algebras associated with scalar second-order ordinary differential equations. *J. Math. Phys.*, 30, No. 12:2770–2777, 1989.
- [128] M. Mathisson. Le problème de M.Hadamard relatif à la diffusion des ondes. *Acta Math.*, 71:249–282, 1939.
- [129] J. A. McLennan. Conformal invariance and conservation laws for relativistic wave equations for zero rest mass. *Nuovo Cimento*, 3:1360, 1956.
- [130] V. M. Men'shikov. Continuous conjugation of invariant solutions. *Continuum Dynamics*, 10:70–84, 1972. Publisher: Institute of Hydrodynamics, USSR Acad. Sci., Siberian Branch, Novosibirsk. (Russian).
- [131] R. M. Miura, editor. *Bäcklund Transformations, the inverse scattering method, solitons, and their applications*. Lecture Notes in Math., 515, Springer Verlag, Berlin and New York, 1976.
- [132] K. Nishijima. *Fundamental particles*. Benjamin, New York, 1963. Russian transl., Moscow, 1965.

- [133] E. Noether. Invariante Variationsprobleme. *Königliche Gesellschaft der Wissenschaften zu Göttingen, Nachrichten. Mathematisch-Physikalische Klasse*, Heft 2:235–257, 1918. English transl., *Transport Theory and Statistical Physics*, vol. 1, No. 3, 1971, 186–207.
- [134] P. J. Olver. *Applications of Lie groups to differential equations*. Springer-Verlag, New York, 1986. 2nd ed., 1993.
- [135] L. V. Ovsyannikov. New solution for equations of hydrodynamics. *Dokl. Akad. Nauk SSSR*, 111, No. 1:47, 1956. (Russian).
- [136] L. V. Ovsyannikov. Group properties of the equation of S.A. Chaplygin. *Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki*, No. 3:126–145, 1960. English transl. in: "Lie group analysis: classical heritage", ed. N.H. Ibragimov, ALGA publications, Karlskrona, Sweden, 2004, pp.123–154.
- [137] L. V. Ovsyannikov. On determination of a group for a second-order linear differential equation. *Dokl. Akad. Nauk SSSR*, 132, No. 1, 1960. (Russian).
- [138] L. V. Ovsyannikov. *Group properties of differential equations*. Siberian Branch, USSR Academy of Sciences, Novosibirsk, 1962. (Russian).
- [139] L. V. Ovsyannikov. *Lectures on the theory of group properties of differential equations*. Novosibirsk Univ. Press, Novosibirsk, 1966. (Russian).
- [140] L. V. Ovsyannikov. Partial invariance. *Dokl. Akad. Nauk SSSR*, 186, No. 1:22–25, 1969. English transl., *Soviet Math. Dokl.*, 10, No. 3, (1969), pp. 538–541.
- [141] L. V. Ovsyannikov. Nonlinear Cauchy problem in a scale of Banach spaces. *Dokl. Akad. Nauk SSSR*, 200, No. 4:789–792, 1971. English transl., *Soviet Math. Dokl.*, 12, No. 5, (1971), p. 1497–1502.
- [142] L. V. Ovsyannikov. *Analytical groups*. Novosibirsk Univ. Press, Novosibirsk, 1972. (Russian).
- [143] L. V. Ovsyannikov. Some problems arising in group analysis of differential equations. *Proc. Sympos. Symmetry, Similarity and Group Theoretic Methods in Mechanics*, Ed. P.G. Glockner and M.C. Singh, University of Calgary, Calgary:181–202, 1974.

- [144] L. V. Ovsiyannikov. *Group analysis of differential equations*. Nauka, Moscow, 1978. English transl., ed. W.F. Ames, Academic Press, New York, 1982. See also L. V. Ovsiyannikov, *Group properties of differential equations*, Siberian Branch, USSR Academy of Sciences, Novosibirsk, 1962.
- [145] W. Pauli. Über die Invarianz der Dirac'schen Wellengleichungen gegenüber Ähnlichkeitstransformationen des Linienelements im Fall verschwindender Ruhmasse. *Helvetica Physica Acta*, 13:204–208, 1940.
- [146] W. Pauli. On the conservation of the lepton charge. *Nuovo Cimento*, 6, No. 1:204–215, 1957.
- [147] R. Penrose. Conformal treatment of infinity. *Rélativité, groupes et topologie*, pages 565–584, 1964. Les Houches Lectures, 1963 Summer School of Theor. Phys., Univ. Grenoble.
- [148] A. Z. Petrov. *Einstein spaces*. Fizmatgiz, Moscow, 1961. Or see his *New methods in the general theory of relativity*, Nauka, Moscow, 1966.
- [149] A. Z. Petrov. *New methods in the general relativity*. Nauka, Moscow, 1966. English transl. *Einstein Spaces*, Pergamon Press, New York, 1969.
- [150] I. G. Petrovsky. *Lectures on partial differential equations*. Fizmatgiz, Moscow, 3 edition, 1961. English transl., Interscience, New York, 1964. Republished by Dover, 1991. Translated from Russian by A. Shenitzer.
- [151] E. Pinney. The nonlinear differential equation $y'' + p(x)y + Cy^{-3} = 0$. *Proc. Am. Math. Soc.*, 1, No. 5:681, 1950.
- [152] V. V. Pukhnachev. Invariant solutions of the Navier-Stokes equations describing dynamics in a free boundary media. *Dokl. Akad. Nauk SSSR*, 202, No. 2:302–305, 1972.
- [153] G. F. B. Riemann. Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. *Abhandl. Königl. Ges. Wiss. Göttingen*, 8:43–65, 1860.
- [154] J. F. Ritt. *Differential Algebra*. AMS Coll. Publ. 33, New York, 1950. Reprinted by Dover, 1966.
- [155] C. Rogers and W. F. Ames. *Nonlinear boundary value problems in science and engineering*. Academic Press, Boston, 1989.

- [156] G. Rosen. Nonlinear heat conduction in solid H_2 . *Phys. Rev.*, B19, No. 4:2398–2399, 1979.
- [157] G. Rumer. Krümmungswellen in der allgemeinen Relativitätstheorie. *Zeitschrift für Physik, A Hadrons and Nuclei*, 171, No. 1:123 – 128, 1963.
- [158] L. Schwartz. *Théorie des Distributions*. t.I, II. Hermann, Paris, 1950-1951.
- [159] L. I. Sedov. *Similarity and dimensional methods in mechanics*. GITTL, Moscow, 1957. English transl., *Similarity and dimensional analysis*, Academic Press, New York, 1959.
- [160] J. Serrin. *Mathematical principles of classical fluid mechanics*. Handbuch der Physik, Band VIII/1, Strömungsmechanik I, Berlin-Göttingen-Heidelberg, 1959.
- [161] V. N. Shapovalov. On group properties of linear equations. *Izvestia Vuzov: Fizika*, No. 6:75, 1968. (Russian).
- [162] V. N. Shapovalov. Symmetry of differential equations, I, II. *Izvestia Vuzov: Fizika*, No. 6:57–64, 64–70, 1977. English transl., *Soviet Phys. J.*, 20, (1977).
- [163] K. L. Stellmacher. Ein Beispiel einer Huygensschen Differentialgleichung. *Nachr. Akad. Wiss., Göttingen, Math.-Phys. Kl. IIa*, 10:133–138, 1953.
- [164] K. L. Stellmacher. Eine Klasse Huygensscher Differentialgleichungen und ihre Integration. *Mathematische Annalen*, 130, No. 3:219–233, 1955.
- [165] H. Stephani. *Differential equations: Their solution using symmetries*. Cambridge Univ. Press, Cambridge, 1989.
- [166] J. J. Stoker. *Water waves: The mathematical theory and applications*. Interscience, New York, 1953.
- [167] A. Stubhaug. *Det var mine tankers djervhet – Matematikeren Sophus Lie*. H.Aschehoug, Oslo, 2000. In Norewegian. English transl., Arild Stubhaug, *The mathematician Sophus Lie: It was the audacity of my thinking*, Springer-Verlag, Berlin, 2002.

- [168] G. L. Synge. *Relativity: The general theory*. North-Holland, Amsterdam, 1964.
- [169] A. H. Taub. A characterization of conformally flat spaces. *Bull. Amer. Math. Soc.*, 55, No. 2:85–89, 1949.
- [170] O. Tedone. Sull'integrazione dell'equazione $\frac{\partial^2 \varphi}{\partial t^2} - \sum_{i=1}^m \frac{\partial^2 \varphi}{\partial x_i^2} = 0$. *Ann. di Mat. Pura Appl.*, Ser III^a, 1:1–23, 1898.
- [171] E. D. Terentyev and Ju. D. Shmyglevskii. A complete system of equations in divergence form for the dynamics of an ideal gas. *Zh. Vychisl. Mat. i Mat. Fiz.*, 15, No. 6:1535–1544, 1975. English transl., USSR Comput. Math. and Math. Phys., 15, (1975).
- [172] A. N. Tikhonov and A. A. Samarskii. *Equations of mathematical physics*. Nauka, Moscow, 5th edition, 1977. English transl., Equations of mathematical physics, Macmillan, New York, 1963.
- [173] H. Umezawa. *Quantum field theory*. North-Holland, Amsterdam, 1956.
- [174] E. Vessiot. Sur une classe d'équations différentielles. *Ann. Sci. Ecole Norm. Sup.*, 10:53, 1893.
- [175] A. M. Vinogradov. Multivalued solutions and a principle of classification of nonlinear differential equations. *Dokl. Akad. Nauk SSSR*, 210, No. 1:11–14, 1973. English transl., Soviet Math. Dokl. 14, (1973).
- [176] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Univ. Press, Cambridge; Macmillan, New York, 2nd edition, 1944.

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SELECTED WORKS

Volume I



Nail H. Ibragimov was educated at Moscow Institute of Physics and Technology and Novosibirsk University and worked in the USSR Academy of Sciences. Since 1976 he lectured intensely all over the world, e.g. at Georgia Tech in USA, Collège de France, University of Witwatersrand in South Africa, University of Catania in Italy, etc. Currently he is Professor of Mathematics and Director of ALGA at the Blekinge Institute of Technology, Karlskrona, Sweden. His research interests include Lie group analysis of differential equations, Riemannian geometry and relativity, mathematical modelling in physics and biology. He was awarded the USSR State Prize in 1985 and the prize *Researcher of the year* by Blekinge Research Society, Sweden, in 2004. N.H. Ibragimov has published 14 books including two graduate textbooks *Elementary Lie group analysis and ordinary differential equations* (1999) and *A practical course in differential equations and mathematical modelling* (2005)

Volume I contains papers written during 1966-1992. The main topics in this volume include Lie group analysis of differential equations, generalized motions in Riemannian spaces, Huygens' principle, theory of Lie-Bäcklund transformation groups, symmetries in fluid dynamics and mathematical physics.



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